PARAMETRIC AMPLIFIERS

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This paper introduces parametric amplification through the basic theory of parametric resonance and then focuses on superconducting implementations. It describes Josephson junctions, the dc-SQUID as a flux-tunable nonlinear element, and the operating principles of Josephson and traveling-wave parametric amplifiers. Finally, it briefly presents optical parametric amplifiers and highlights key applications in quantum measurements and modern spectroscopy.

PARAMETRIČNI OJAČEVALNIKI

Članek obravnava parametrično ojačevanje prek parametrične resonance, nato pa se osredotoči na implementacijo samih ojačevalcev. Opiše Josephsonove spoje, enosmerni SQUID kot nelinearni element, nastavljiv z magnetnim pretokom, ter načela delovanja Josephsonovih in traveling-wave parametričnih ojačevalnikov. Na koncu na kratko predstavi optične parametrične ojačevalnike in izpostavi ključne uporabe pri kvantnih meritvah in sodobni spektroskopiji.

1. Introduction

Parametric amplifiers are a special class of amplifiers in which energy is transferred from a strong pump signal to a weaker signal through periodic modulation of a system parameter, such as capacitance or inductance. Unlike many conventional amplifiers, they can provide large gain while adding very little noise, which makes them especially valuable in experiments where signals are extremely weak.

This is why parametric amplification is an important topic in modern physics and engineering. In superconducting quantum circuits, parametric amplifiers are essential for high-fidelity qubit readout. In fundamental physics, they are used in sensitive searches for weak signals, such as dark-matter candidates.

This paper overviews parametric amplifiers and their theoretical background. First, I introduce parametric resonance and derive the conditions under which periodic parameter modulation leads to amplification. Next, I present superconducting implementations, starting with Josephson junctions and the dc-SQUID as a flux-tunable nonlinear element, and then describe the operating principles of Josephson parametric amplifiers. I then discuss traveling-wave parametric amplifiers and optical parametric amplifiers, with emphasis on practical applications. Finally, I summarize recent research advances and their relevance for quantum measurement and readout technologies.

2. Parametric resonance

To see how parametric amplifiers enhance the signal, we must first understand the concept of parametric resonance. Parametric resonance occurs in oscillatory systems which are not closed, but in which the external action amounts only to a time variation of the parameters. The simplest oscillatory system is described by the equation of motion

$$\ddot{x} + \omega_0^2 x = 0, \tag{1}$$

where ω_0 is the natural frequency of the system. If we now make the frequency in (1) time dependent, we get

$$\ddot{x} + \omega^2(t)x = 0. \tag{2}$$

If we are interested in a more general case with a damping term

$$\ddot{x} + \beta(t)\dot{x} + \omega^2(t)x = 0,$$

where $\beta(t)$ is the time-dependent damping coefficient, we can make the substitution

$$x(t) \to \exp\left(-\tfrac{1}{2}\int \beta(t)\,dt\right)x(t), \tag{3}$$

to transform the equation into the form of (2) with a modified

$$\omega^2(t) \to \omega^2(t) - \frac{1}{2}\frac{d\beta(t)}{dt} - \frac{1}{4}\beta^2(t), \tag{4}$$

which includes the effect of damping. This shows that we can analyze Equation (2) without loss of generality, as it can be transformed to include damping effects as well [1].

Now we follow the analysis of Landau and Lifshitz [2] to understand the solutions of Equation (2). Let us assume that $\omega(t)$ is periodic with period $T = 2\pi/\gamma$. This means $\omega(t) = \omega(t+T)$, and Equation (2) is invariant under the transformation $t \to t + T$. Hence, if $x(t)$ is a solution of the equation, so is $x(t+T)$. If there exist two independent solutions $x_1(t)$ and $x_2(t)$, they must be summed to get the general solution.

Due to Floquet's theorem, the solutions of a linear differential equation with a periodic system matrix $A(t+T) = A(t)$, for a system $\dot{\mathbf{x}} = A(t)\mathbf{x}$, are multiplied by a constant factor after one period T, i.e. $x_1(t+T) = \mu_1 x_1(t)$ and $x_2(t+T) = \mu_2 x_2(t)$ [3]. The most general functions to satisfy this property are

$$x_1(t) = \mu_1^{t/T}\Pi_1(t), \quad x_2(t) = \mu_2^{t/T}\Pi_2(t), \tag{5}$$

where Π_1 and Π_2 are purely periodic functions of time with period T.

The constants μ_1 and μ_2 obey constraints that can be derived from the equations $\ddot{x}_1 + \omega^2(t)x_1 = 0$ and $\ddot{x}_2 + \omega^2(t)x_2 = 0$. If we multiply the first one by x_2, the second one by x_1 and subtract them, we get $\frac{d}{dt}\left(\dot{x}_1 x_2 - x_1\dot{x}_2\right) = 0$, or

$$\dot{x}_1 x_2 - x_1 \dot{x}_2 = \text{const.} \tag{6}$$

For any functions $x_1(t)$ and $x_2(t)$ of the form (5), the expression on the left-hand side of (6) is multiplied by $\mu_1\mu_2$ when $t \to t + T$. Hence, the following must hold:

$$\mu_1\mu_2 = 1. \tag{7}$$

Further information about the constants μ_1 and μ_2 can be obtained from the fact that the coefficients in Equation (2) are real. If $x(t)$ is a solution of such an equation, then the complex conjugate function $x^*(t)$ must also be a solution. Hence it follows that μ_1 and μ_2 must be the same as μ_1^* and μ_2^*, i.e. either $\mu_1 = \mu_2^*$ or μ_1 and μ_2 are both real. In the former case, (7) gives $\mu_1 = 1/\mu_1^*$ and $|\mu_1|^2 = |\mu_2|^2 = 1$.

In the other case, two independent integrals of Equation (2) are given by

$$x_1(t) = \mu^{t/T}\Pi_1(t), \quad x_2(t) = \mu^{-t/T}\Pi_2(t), \tag{8}$$

with a positive or negative real value of μ ($|\mu| \neq 1$). One of the functions in (8) grows exponentially with time. This means that the system at rest in equilibrium $x = 0$ is unstable, i.e. any deviation from this state, however small, is sufficient to lead to a rapidly increasing displacement x. This is called parametric resonance.

It should be noticed that, when the initial values of x and $\dot{x}$ are exactly zero, they remain zero, unlike what happens in ordinary resonance, in which the displacement increases with time (proportionally to t) even from initial values of zero.

Let us now consider the case of parametric resonance when $\omega(t)$ is weakly modulated around a constant value ω_0 and is a simple periodic function

$$\omega^2(t) = \omega_0^2[1 + h\cos(\gamma t)], \tag{9}$$

where the constant $h \ll 1$. We shall suppose h positive, as may always be done by suitably choosing the origin of time. As shown in Figure 1, parametric resonance is strongest if the frequency of the modulation in (9) is nearly twice ω_0. Hence we put $\gamma = 2\omega_0 + \epsilon$, where $\epsilon \ll \omega_0$. The regions of stable and unstable solutions are illustrated in Figure 1. Substituting (9) with $\gamma = 2\omega_0 + \epsilon$ into (2),

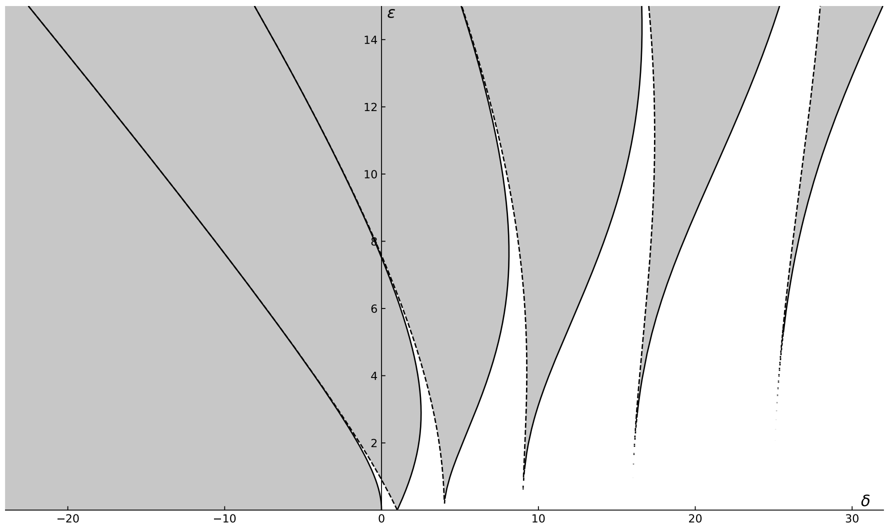

Figure 1. The stable and unstable regions of the Mathieu equation $\ddot{x} + \delta(1 + 2\epsilon/\delta\cos 2t)x = 0$. Variable ϵ in this figure is different from ϵ in the Equation (10). In the figure the frequency of the modulation is set to $\omega_0 = 1$ and the modulation amplitude is set to $h = 2\epsilon$, because we know δ via relationship $\delta = \omega_0^2 = 1$. The shaded regions correspond to unstable solutions, where parametric resonance occurs, which means the amplitude of solution will grow exponentially with time. White area corresponds to stable solutions, where the amplitude will not grow exponentially. The two regions are separated by the boundary curves which are called transition values, in which at least one of the solutions is periodic with the period π or 2π [1]. Solid lines tell us that solution is even and dashed lines tell us that solution is odd [4].

the equation of motion becomes

$$\ddot{x} + \omega_0^2\left[1 + h\cos\left((2\omega_0 + \epsilon)t\right)\right]x = 0 \tag{10}$$

and the solution may be sought in the form

$$x(t) = a(t)\cos\left(\omega_0 + \tfrac{1}{2}\epsilon\right)t + b(t)\sin\left(\omega_0 + \tfrac{1}{2}\epsilon\right)t, \tag{11}$$

where $a(t)$ and $b(t)$ are slowly varying functions of time in comparison to the oscillating terms. This form of solution is, of course, not exact. In reality, the function $x(t)$ also involves terms with frequencies which differ from $\omega_0 + \frac{1}{2}\epsilon$ by multiples of $2\omega_0 + \epsilon$. When one looks at the condition for parametric resonance, it is sufficient to consider only the terms with frequencies close to $\omega_0 + \frac{1}{2}\epsilon$, as the other terms give corrections of higher order $\mathcal{O}(h^2)$.

We substitute the expression (11) into (10) and retain only terms of the first order in ϵ, assuming that $\dot{a} \sim \epsilon a$ and $\dot{b} \sim \epsilon b$; the correctness of this assumption under resonance conditions is confirmed by the result. One can also understand this assumption by noting that the functions $a(t)$ and $b(t)$ are slowly varying, and set their second derivatives to zero. The products of trigonometric functions may be replaced by sums, for example

$$\cos\left(\omega_0 + \tfrac{1}{2}\epsilon\right) t \cos\left((2\omega_0 + \epsilon)t\right) = \tfrac{1}{2}\left[\cos 3\left(\omega_0 + \tfrac{1}{2}\epsilon\right) t + \cos\left(\omega_0 + \tfrac{1}{2}\epsilon\right) t\right], \tag{12}$$

and in accordance with what was said above we omit the terms with frequency $3(\omega_0 + \frac{1}{2}\epsilon)$ in (12), as they give only small corrections. The result is

$$-\left(2\dot{a} + b\epsilon + \tfrac{1}{2}h\omega_0 b\right)\sin\left(\omega_0 + \tfrac{1}{2}\epsilon\right) t + \left(2\dot{b} - a\epsilon + \tfrac{1}{2}h\omega_0 a\right)\cos\left(\omega_0 + \tfrac{1}{2}\epsilon\right) t = 0. \tag{13}$$

If this equation is to hold, the coefficients of the sine and cosine in (13) must both be zero. This gives two linear differential equations for the functions $a(t)$ and $b(t)$. Ansatz is of the form $a(t), b(t) \sim C_1 e^{st} + C_2 e^{-st}$. Constants C_1 and C_2 are given by initial conditions. For example if $x(t=0) = 0$ and $\dot{x}(t=0) = 0$, then $C_1 = 0$ and parametric resonance will not occur, but in the real world there are always some small movements of system so that $C_1 \neq 0$.

If we plug in the ansatz, we get two homogeneous linear equations for the coefficients a and b,

$$\begin{bmatrix} s & \frac{1}{2}\left(\epsilon + \frac{1}{2}h\omega_0\right) \\ -\frac{1}{2}\left(\epsilon - \frac{1}{2}h\omega_0\right) & s \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If the determinant of the matrix is zero, there are nontrivial solutions for a and b, which gives

$$s^2 = \frac{1}{4}\left[\left(\tfrac{1}{2}h\omega_0\right)^2 - \epsilon^2\right]. \tag{14}$$

The condition for parametric resonance is that s is real, i.e. $s^2 > 0$. Thus parametric resonance occurs in the range

$$-\tfrac{1}{2}h\omega_0 < \epsilon < \tfrac{1}{2}h\omega_0, \tag{15}$$

on either side of the frequency $2\omega_0$, as given by (15). The constant μ is related to s by $\mu = -\exp(s\pi/\omega_0)$.

Parametric resonance also occurs when the frequency γ with which the parameter varies is close to any value $2\omega_0/n$, where n is an integer. The width of the resonance range (region of instability) decreases rapidly with increasing n, however, namely as h^n. The amplification coefficient of the oscillations also decreases.

The phenomenon of parametric resonance is maintained in the presence of slight friction, but the region of instability becomes somewhat narrower. Friction results in a damping of the amplitude of oscillations as $\exp(-\lambda t)$. Therefore the amplification of the oscillations in parametric resonance behaves as $\exp((s-\lambda)t)$, with the positive s given by the solution for the frictionless case, and the limit of the region of instability given by the equation $s - \lambda = 0$. Thus, with s given by (14), we have for the resonance range

$$-\sqrt{\left(\tfrac{1}{2}h\omega_0\right)^2 - 4\lambda^2} \;<\; \epsilon \;<\; \sqrt{\left(\tfrac{1}{2}h\omega_0\right)^2 - 4\lambda^2}. \tag{16}$$

It should be noticed that resonance is now possible not for arbitrarily small amplitudes h, but only when h exceeds a threshold value $h_\mathrm{k} = 4\lambda/\omega_0$, as follows from (16). It can be shown that, for resonance near the frequency $2\omega_0/n$, the threshold value h_k is proportional to $\lambda^{1/n}$, i.e. it increases with n.

The phenomenon of parametric amplification can be seen in the real world in something as familiar as a children's swing. If you crouch your legs and extend them twice during every swing cycle, at exactly the right moments, you change the angle of pendulum. Even though you never push the swing directly, this periodic change is enough to make the swing go higher and higher. This is parametric amplification in its simplest form, instead of adding energy by pushing directly, energy is added by periodically changing a property of the system at the right rate and magnitude.

Electronic parametric amplifiers work on exactly the same principle, but with a circuit element instead of your body, i.e. a component whose behavior can be changed by an external signal, called a nonlinear element. An ordinary, linear component always responds the same way no matter how strong the signal is double the input, double the output. A nonlinear element doesn't, its response changes with signal strength.

Many components show some nonlinearity, but most are weak, lossy, or hard to control precisely. The Josephson junction stands out because it behaves as low-loss nonlinear element, and its properties can be tuned easily just by adjusting a current or magnetic field [5]. This simplicity is why the Josephson junction is a building block for a parametric amplifier.

3. Josephson parametric amplifier

3.1 Josephson junctions

One of the most important properties of a superconductor is the Meißner–Ochsenfeld effect, i.e. the expulsion of magnetic flux from the bulk of the superconductor below a certain transition temperature. This effect can be explained by the macroscopic quantum nature of the superconducting state, where, under certain conditions, normal electrons may form a quantum condensate of so-called Cooper pairs with a unified wave function, in other words a global wave function of the superconductor [6, 7]

$$\Psi(\mathbf{r}) = \sqrt{n(\mathbf{r},t)}\, e^{i\theta(\mathbf{r},t)}, \tag{17}$$

where $n(\mathbf{r},t)$ is the density of Cooper pairs and $\theta(\mathbf{r},t)$ is the phase of the wave function. With this fact, B. Josephson theoretically predicted the Josephson effect, where the two wave functions of superconductors overlap. An overlap of the wave functions can be achieved by placing a thin layer of non-superconducting material between the two superconductors, as shown in Figure 2. The resulting structure is called a Josephson junction [6].

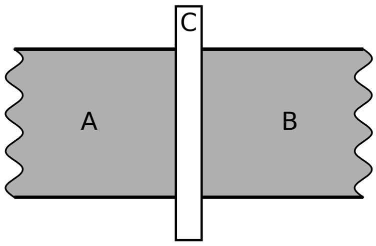

Figure 2. Diagram of a single Josephson junction. The regions A and B represent the two superconductors, and C the thin weak link of non-superconducting material between them.

In order to analyze such a junction we denote the amplitude to find an electron on one side by ψ_{A}, and the amplitude to find it on the other by ψ_{B}. In the superconducting state the wave function ψ_{A} is the common wave function of all the electrons on one side, and ψ_{B} is the corresponding function on the other side. Let us take a very simple situation in which the material is the same on both sides, so that the junction is symmetrical, and let there be no magnetic field for a moment. Then the two amplitudes are related as

$$i\hbar\frac{\partial\psi_{\mathrm{A}}}{\partial t} = U_1\psi_{\mathrm{A}} + K\psi_{\mathrm{B}},$$
$$i\hbar\frac{\partial\psi_{\mathrm{B}}}{\partial t} = U_2\psi_{\mathrm{B}} + K\psi_{\mathrm{A}}.$$

The constant K is a characteristic of the junction that comes from the coupling between the two superconductors. If K were zero, these two equations would just describe the lowest energy state with energy U of each superconductor. Suppose that we connect the two superconducting regions to the two terminals of a battery so that there is a potential difference V across the junction. Then $U_1 - U_2 = qV$, where q is the charge of the Cooper pair. We define the zero of energy to be halfway between, so that $U_1 = +qV/2$ and $U_2 = -qV/2$. Now we use the substitutions $\psi_\mathrm{A} = \sqrt{n_\mathrm{A}}\exp(i\theta_\mathrm{A})$ and $\psi_\mathrm{B} = \sqrt{n_\mathrm{B}}\exp(i\theta_\mathrm{B})$ from (17). We obtain four equations by equating the real and imaginary parts in each case. Letting $\varphi = \theta_\mathrm{B} - \theta_\mathrm{A}$ and writing the equations compactly, we get

$$\dot{n}_\mathrm{A} = -\dot{n}_\mathrm{B} = \frac{2K}{\hbar}\sqrt{n_\mathrm{A} n_\mathrm{B}}\sin\varphi, \tag{18}$$

$$\dot{\varphi} = \dot{\theta}_\mathrm{B} - \dot{\theta}_\mathrm{A} = \frac{qV}{\hbar}. \tag{19}$$

Here it must be mentioned that in the presence of an external magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$, the phase difference across the Josephson junction is given by $\varphi = \theta_\mathrm{B} - \theta_\mathrm{A} - \frac{2\pi}{\Phi_0}\int_\mathrm{A}^\mathrm{B} \mathbf{A}\cdot d\mathbf{l}$, where $\mathbf{A}$ is the vector potential and $\Phi_0 = h/2e$ is the magnetic flux quantum and h is Planck's constant [6].

The first equation, (18), gives the current through the junction. Such a current would charge up one side, except that we keep the potential difference V constant through the battery. Equations (18) and (19) can be solved analytically if we assume a constant number of Cooper pairs $n_\mathrm{A} + n_\mathrm{B} = N$ and a constant voltage. The solution for the current is periodic,

$$I(\varphi) = I_\mathrm{c}\sin\varphi,$$

where I_c is the critical current of the junction. In this context, it is useful to define the nonlinear inductance

$$L_\mathrm{s} = -V\left(\frac{dI}{dt}\right)^{-1} = \frac{\Phi_0}{2\pi I_\mathrm{c}\cos\varphi}.$$

The nonlinear properties of a Josephson junction make it a central building block for superconducting circuits, where it is often utilized as a nonlinear, lossless inductance [7].

3.2 dc-SQUID

A central building block of the JPA (Josephson parametric amplifier) is a direct-current superconducting quantum interference device (dc-SQUID). It consists of two Josephson junctions carrying currents I_1 and I_2 in a superconducting loop, as shown in Figure 3.

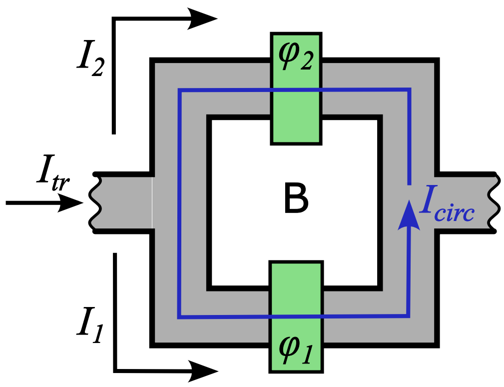

Figure 3. Diagram of a dc-SQUID. The transport current I_tr enters and splits into the two paths, each carrying current I_1 and I_2. The thin green barriers are insulators, which together separate the two superconducting regions colored gray. The vector $\mathbf{B}$ represents the magnetic field threading the dc-SQUID loop.

The phase change of a wave function along a closed loop around any superconducting region must be an integer multiple of 2π, due to the fact that the wave function depends only on position and not on the path taken. This can be written mathematically as

$$\oint \nabla\theta \cdot d\mathbf{s} = 2\pi n,$$

where $n \in \mathbb{Z}$ and θ is the phase of the wave function. Now let us look at the phase change of a wave function along the path of the current I_1 in Figure 3. We can write it as

$$\Delta\theta = \varphi_1 + \frac{q}{\hbar}\int\limits_{\text{lower}} \mathbf{A} \cdot d\mathbf{s}.$$

Similarly, for the upper path with current I_2, we have

$$\Delta\theta = \varphi_2 + \frac{q}{\hbar}\int\limits_{\text{upper}} \mathbf{A} \cdot d\mathbf{s}.$$

The difference in phases must be the same no matter which path we take. Equating the two expressions yields

$$\varphi_1 - \varphi_2 = \frac{q}{\hbar}\oint \mathbf{A} \cdot d\mathbf{s} = \frac{q}{\hbar}\iint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{S} = \frac{2\pi}{\Phi_0}\Phi, \tag{20}$$

where $\Phi = \Phi_{\text{ext}} + \Phi_{\text{ind}}$ is the total magnetic flux threading the dc-SQUID loop, which consists of the externally applied flux Φ_{ext} and the self-induced flux $\Phi_{\text{ind}} = L_{\text{loop}} I_{\text{circ}}$ due to the circulating current in the loop. Now we define the currents

$$I_1 = I_{\text{c}} \sin\varphi_1,$$
$$I_2 = I_{\text{c}} \sin\varphi_2,$$

where I_{c} is the critical (maximal) current of the junctions. It is useful to define the total transport current through the dc-SQUID,

$$I_{\text{tr}} = I_1 + I_2 = 2I_{\text{c}} \sin\left(\frac{\varphi_1 + \varphi_2}{2}\right)\cos\left(\frac{\varphi_1 - \varphi_2}{2}\right), \tag{21}$$

which is given by the sum of the currents through each Josephson junction, as in (21). The circulating current in the loop, which gives the self-induced flux $\Phi_{\text{ind}} = L_{\text{loop}} I_{\text{circ}}$, is

$$I_{\text{circ}} = \frac{I_1 - I_2}{2} = -I_{\text{c}} \cos\left(\frac{\varphi_1 + \varphi_2}{2}\right)\sin\left(\frac{\varphi_1 - \varphi_2}{2}\right). \tag{22}$$

The self-induced flux is usually small compared to the external flux, so we can neglect it when $L_{\text{loop}} I_{\text{circ}} \ll \Phi_{\text{ext}}$; otherwise we would need to solve equations that have no analytical solutions [7]. The circulating current (22) thus represents only a small correction. We now define the flux-tunable inductance of the dc-SQUID,

$$L_{\text{squid}}(\Phi_{\text{ext}}) = \frac{\Phi_0}{2\pi I_{\text{tr}}^{\max}} = \frac{\Phi_0}{4\pi I_{\text{c}} \left|\cos\left(\pi \frac{\Phi_{\text{ext}}}{\Phi_0}\right)\right|}. \tag{23}$$

From (23) we see that the dc-SQUID can be applied as a nonlinear element in superconducting circuits shown in Figure 4, where the inductance can be tuned by the external magnetic flux Φ_{ext}, and the nonlinearity enables parametric resonance in the JPA circuit [7].

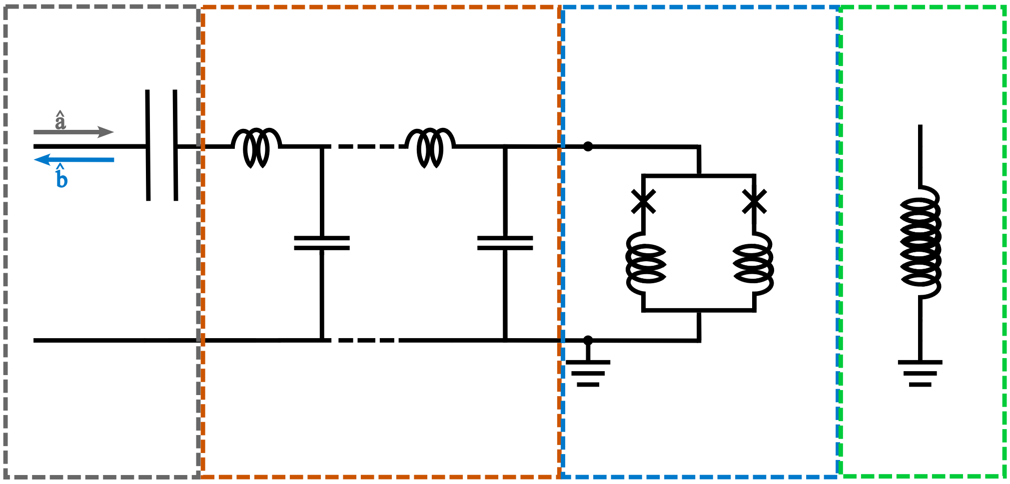

Figure 4. Circuit diagram of a flux-driven JPA consisting of an input capacitance (gray), resonator (orange), dc-SQUID (blue), and pump line (green). The crossed lines represent Josephson junctions [7].

4. Parametric amplification

In general, amplification in nonlinear circuits can be enabled by driving a parametric process with an external pump tone (for example $\Phi_{\rm ext} \to \Phi_{\rm ext}(t)$). The pump tone must vary one of the parameters of the resonator, such as the inductance L or capacitance C of the circuit, in a periodic fashion that gives rise to parametric effects. In principle, we do not need a nonlinear element to have a parametric process, but in practice it is preferable to have one, because it allows the detected signals to have higher frequencies. For example if we had two electrodes that made up a capacitor, we could vary the distance between them to change the capacitance. However, this would be difficult to do at high frequencies i.e. GHz, and it would be hard to control the process, because of mechanical limits.

For superconducting circuits, Josephson junctions are routinely used for this purpose. In a flux-driven JPA, the pump tone is inductively coupled to the circuit and is driven at a frequency $\omega_{\rm p}$, chosen to be roughly twice the resonance frequency ω_0 of the JPA circuit, i.e. $\omega_{\rm p} \approx 2\omega_0$. This pump tone leads to a periodic modulation of the dc-SQUID inductance which, in turn, causes a periodic modulation of the resonance frequency ω_0 itself, as shown in Figure 5 [7].

Consequently, the induced parametric modulation enables a three-wave mixing process in which an incident signal mode at frequency $\omega_{\rm s} = \omega_{\rm p}/2 + \delta\omega$, with detuning $\delta\omega$, is amplified, as depicted in Figure 5. At the same time, an idler mode at frequency $\omega_{\rm i} = \omega_{\rm p}/2 - \delta\omega$ is created [7].

To understand why this happens, we look at the equation of motion of the JPA circuit, which has the form (10), where the variable x represents the current I. If we force the system to behave at $t = 0$ as

$$x(t) = a(t)\cos(\omega_{\rm s} t) + b(t)\sin(\omega_{\rm i} t), \tag{24}$$

where $a(t)$ and $b(t)$ are the slowly varying amplitudes of the readout signal, so that we can neglect their second derivatives, and $\omega_{\rm s} = \omega_0 + \delta\omega$ and $\omega_{\rm i} = \omega_0 - \delta\omega$. If we plug the expression (24) into the equation of motion, we get two coupled equations for $a(t)$ and $b(t)$. The starting conditions are $a(0) = a_0$ and $b(0) = 0$, because only the first term represents the input signal. The solution of these equations is that both a and b grow exponentially with time, which means that the first term is amplified and the idler term is created. If $\delta\omega = 0$, the signal and idler frequencies are the same and we have degenerate parametric amplification; otherwise, if $\delta\omega \neq 0$, we have non-degenerate parametric amplification [1, 7].

One can also think of this process as a conversion of pump photons (or energy) into signal and

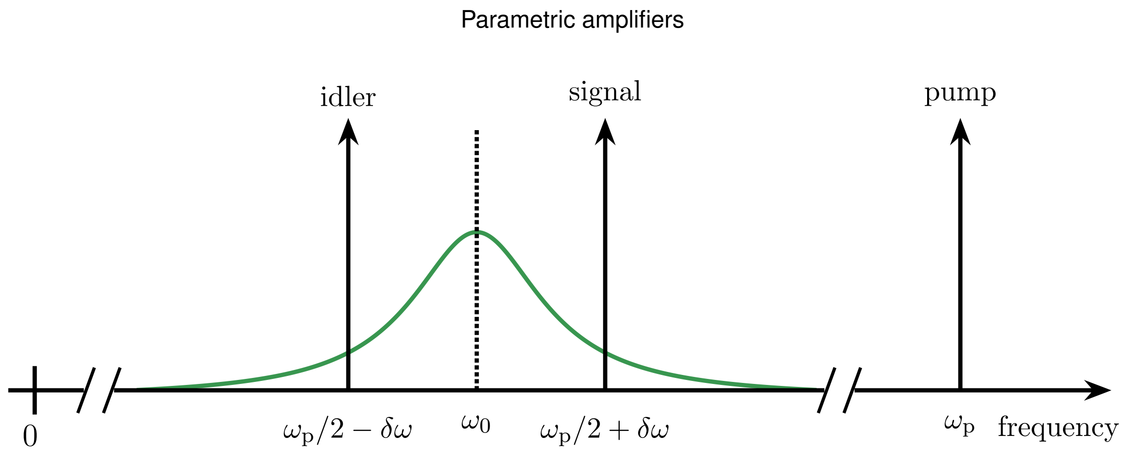

Figure 5. Scheme of the relevant frequencies for parametric amplification with a flux-driven JPA. The pump frequency ω_p is at roughly twice the JPA resonance frequency ω_0. This kind of amplification is called non-degenerate parametric amplification, as the signal and idler frequencies are not the same [7].

idler photons, $\omega_\mathrm{p} = \omega_\mathrm{s} + \omega_\mathrm{i}$, which is the reason this process is called three-wave mixing. The pump tone provides the energy for the amplification, and the nonlinearity of the circuit allows the mixing of frequencies that leads to amplification of the signal and generation of the idler [7].

A more fundamental explanation of the three-wave mixing process follows from the nature of quantum mechanics. The input signal can be described by operators $\hat{a}$ and $\hat{a}^\dagger$, which satisfy the usual commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. Similarly, we have output operators $\hat{b}$ and $\hat{b}^\dagger$ that satisfy the bosonic commutation relation. If the output signal is amplified by a factor G, we can write the relation between output and input as $\hat{b} = \sqrt{G}\,\hat{a}$, but this violates the bosonic commutation relation, because $[\hat{b}, \hat{b}^\dagger] = G$. We are therefore forced to write $\hat{b} = \sqrt{G}\,\hat{a} + \hat{\mathcal{F}}$, where $\hat{\mathcal{F}}$ is an operator representing the additional noise added by the amplifier. Because the noise is uncorrelated with the input signal, $\langle \hat{\mathcal{F}}\hat{a} \rangle = \langle \hat{\mathcal{F}}\hat{a}^\dagger \rangle = 0$, we can derive the commutation relation for the noise, $[\hat{\mathcal{F}}, \hat{\mathcal{F}}^\dagger] = 1 - G$. It turns out that the idler mode is created due to the noise operator $\hat{\mathcal{F}}$ [8].

4.1 Traveling-wave parametric amplifiers

The first parametric amplifier was demonstrated in the 1960s by Louisell, Yariv, and Siegman, who developed the theory and performed the first experiments. These amplifiers were based on nonlinearities in varactor diodes (also called varicap, variable-capacitance, variable-reactance, or tuning diodes), which are semiconductor devices whose capacitance depends on the applied voltage through a p–n junction [9, 10]. Only later, in the 1980s, was the first JPA demonstrated by Yurke and colleagues, and since then such devices have been widely used in various applications [11].

One of the most important applications of parametric amplifiers is in quantum information processing, where they operate at millikelvin temperatures. The microwave signals from qubits are extremely weak, at the single-photon level, and must be amplified without adding thermal noise that would destroy the quantum state. For the readout we use traveling-wave parametric amplifiers (TWPAs) which, in contrast to Josephson parametric amplifiers (JPAs), exhibit a broad bandwidth that is not constrained by resonator limitations. This allows TWPAs to read out multiple qubits simultaneously [12].

TWPAs are made out of a long chain of Josephson junctions or SQUIDs, forming a transmission line and does not have the fixed gain-bandwidth intrinsic to resonant JPAs. When a strong microwave pump wave propagates down the line, the nonlinear inductance of the junctions permits wave mixing [13, 14].

TWPAs have also been used to search for galactic axions, which are hypothetical particles that could make up dark matter [15]. The idea is that axions can convert into photons in the presence of a strong magnetic field, and TWPAs can detect these photons with high sensitivity. They are also used to detect electron spin resonance [16] and to sense the motion of nanomechanical resonators [17].

4.2 Optical parametric amplifiers

There are also optical parametric amplifiers (OPAs), which enable precise light amplification and wavelength tuning through nonlinear optical processes. OPAs have contributed new scientific techniques, such as time-resolved spectroscopy and fluorescence measurements, which advance applications in biochemistry and biology as diverse as protein folding, tumor-cell invasion, and neuron signaling, all of which require light tuned to very precise wavelengths [18].

5. Latest research

In recent work from 2025 [19], a research team at RIKEN created a new traveling-wave design: by dropping the use of lossy dielectric material and instead fabricating a spiralled, fishbone-like, gradually narrowing waveguide structure, the team was able to reduce the noise close to the quantum limit.

A measurable improvement in multi-qubit readout was achieved using a kinetic-inductance traveling-wave parametric amplifier (KI-TWPA). Previously, KI-TWPAs fabricated from a 20 nm-thick film had been operated near the quantum limit; however, these devices required a pump-tone amplitude too large for adequate temperature control during high-precision qubit readout. Reducing the film thickness to 10 nm successfully lowered the required pump power [20].

There has also been a demonstration of single-atom parametric amplifiers (SAPAs), which can be used to enhance qubit readout, as evidenced by a more than twofold improvement in the signal-to-noise ratio [21].

6. Conclusion

In this paper we have discussed the phenomenon of parametric resonance in oscillatory systems with time-varying parameters. We have derived the conditions for parametric resonance and explored its implications in various physical systems. We have also examined the application of parametric resonance in Josephson parametric amplifiers (JPAs), which are crucial components in superconducting circuits for quantum information processing. Finally, we have highlighted some of the latest research advances in the field of parametric amplification, showcasing the ongoing efforts to enhance the performance and capabilities of these amplifiers.

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