COHOMOLOGICAL LOWER BOUND ON THE LUSTERNIK–SCHNIRELMANN CATEGORY OF A SMALL CATEGORY

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To each topological space X, one can assign the least cardinality of an open cover with each member contractible inside X. If this cardinality is finite, then the related number is called the Lusternik–Schnirelmann category of X. Recently, Kohei Tanaka modified the notion of Lusternik–Schnirelmann category (still a number) so that it is an invariant of (actual) small categories. A classical theorem says that the Lusternik–Schnirelmann category of a topological space can be bounded from below by its cup-length (a number determined by its singular cohomology ring). In the present paper, an analogous result is proved, where X is a small category. On the way to this result, an exposition of the nerve construction and cohomology of a small category is given, and the functoriality and „homotopy“ invariance of this cohomology are proved.

KOHOMOLOŠKA SPODNJA MEJA ZA LUSTERNIK–SCHNIRELMANNOVO ŠTEVILO MAJHNE KATEGORIJE

Za topološki prostor X se poraja vprašanje: kakšna je najmanjša kardinalnost odprtega pokritja X, za katerega velja, da je vsaka odprta množica kontraktibilna znotraj X? Kadar je ta kardinalnost končna, se ustrezno število imenuje Lusternik–Schnirelmannova kategorija X. Kohei Tanaka je pojem Lusternik–Schnirelmannove kategorije (število) priredil, tako da se aplicira na (dejanske) majhne kategorije. Klasični izrek pravi, da je Lusternik–Schnirelmannovi kategoriji topološkega prostora X mogoče postaviti spodnjo mejo – kupa-dolžino (angl. cup-length), definirano preko singularne kohomologije X. V pričujočem članku je dokazan analogni izrek, kjer je X majhna kategorija. Na poti do tja so predstavljeni pojmi živca, kohomologije majhne kategorije, funktorialnosti in „homotopske“ invariantnosti te kohomološke konstrukcije.

1. Introduction

Let X be a topological space. The Lusternik–Schnirelmann category of X is defined as the lowest possible cardinality of a finite open covering of X with the property that each member of the covering is contractible inside[1] of X. The notion has great importance in relation to the area of topological complexity [1]. The notion of Lusternik–Schnirelmann category has then been modified by Kohei Tanaka [2] so that it applies to small categories instead of topological spaces. A flurry of results followed [3, 4, 5] which treat small categories analogously to topological spaces, establishing categorical versions of classical theorems of classical topology. The present paper is concerned with establishing one such categorical analogue of a classical topological theorem. The topological theorem says that for a topological space X, the cup-length of X, defined via the singular cohomology of X, is smaller than the Lusternik–Schnirelmann category of X. Our goal in this article is two-fold. One is our main theorem: if X is a connected small category and there exists a finite geometric cover of X with subcategories contractible in X, then the cup-length $\mathrm{cl}(X)$ is finite and $\mathrm{cl}(X) < \mathrm{cat}(X)$. We will define the cup-length $\mathrm{cl}(X)$ in analogy with the definition of cup-length for topological spaces (indeed the two definitions are related to a certain degree). Of course, for that to work, a cohomology theory of small categories has to be in place. So the second goal of this paper is to present how a cohomological construction for small categories can work.

The organisation of this paper is as follows. In section 2. we give an introduction to simplicial sets and our cohomological construction. In section 3. we show that for each small category X, the resulting cohomological construction on X gives a graded commutative ring, analogously to the

[1] A subset U of X is contractible inside X, if the inclusion map $U \hookrightarrow X$ is nullhomotopic.

situation for singular cohomology (indeed the latter is a special case of the former). In section 4. we show that our cohomological construction is functorial, and that the resulting functor enjoys a nice property reminiscent of homotopy invariance of singular cohomology. Finally, in section 5. we define the cup-length of a small category X and prove our main theorem. Although this paper presents a result proved by the author independently (and later extended [2]) it is also an expository essay, especially in the first half.

2. Simplicial sets and cohomology of small categories

Our aim in this section is to introduce simplicial sets, present the classical construction of cohomology of simplicial sets, introduce the notion of a nerve of a small category, and finally construct the cohomology groups of a small category by piecing together the constructions. This circle of ideas is sometimes considered abstract, but what we want to show here is that these ideas are neighboring on very classical notions from algebraic topology.

2.1 Simplicial sets

Simplicial sets are combinatorial objects, devised as a representation of homotopy types. Technically, they are constructed as presheaves of sets on a simplex category Δ, which we are about to introduce. The intuition should already be pretty clear to categorical enthusiasts. Since presheaves are colimits of representable presheaves in a canonical way, and since colimits are intuitively something like gluing, we may view an arbitrary simplicial set as a gluing of objects of the simplex category. Therefore, if the objects of the simplex category Δ are precisely simplices, then a simplicial set will have no choice but to be a formal gluing of simplices.

Definition 1. We define the simplex category Δ so that:

1. Objects of Δ are finite linearly ordered sets $[n] = \{0 < 1 < \cdots < n\}$, for integers $n \geq 0$.
2. A morphism in Δ is a monotone map $f : [n] \to [m]$. In other words, $f : [n] \to [m]$ has the property that $0 \leq i \leq j \leq n$ implies $f(i) \leq f(j)$.

It is quickly seen that Δ is indeed a category with composition of morphisms defined in the expected way.

Why would this be viewed as a category of simplices? We imagine each partially ordered set $[n]$, with $n \geq 0$, as a simplex with ordered vertices. For each injective map $f : [n] \to [m]$, we imagine we have an inclusion of the n-simplex into the m-simplex as a face (or a subface – a face of a face ...). For a general morphism $f : [n] \to [m]$ in Δ, we imagine a map from the n-simplex to the m-simplex, which first performs a projection of the n-simplex into a possibly lower dimensional simplex, and then embeds it into the m-simplex as a face (or a subface ...). In this way, Δ records the information of maps between simplices, where the simplices have ordered vertices, and maps respect these orderings (although they are not necessarily injective).

For integers $n \geq 1$ and $0 \leq j \leq n$, define a morphism of Δ, $\delta_j^n : [n-1] \to [n]$ with

$$\delta_j^n(x) = \begin{cases} x, & \text{if } x < j \\ x+1, & \text{if } x \geq j. \end{cases}$$

Usually, we will write δ_j instead of δ_j^n, since the upper index n will be apparent from context. The geometric interpretation of the map δ_j^n is that this is the jth face of the n-simplex (that is, the face that is opposite to the jth vertex of the n-simplex).

Definition 2. Simplicial sets are cofunctors from the simplex category Δ to the category of sets (i.e. presheaves on Δ). The value of a simplicial set X on $[n]$ will be denoted X_n, for all nonnegative integers $n \geq 0$.

As we said, simplicial sets were initially devised as combinatorial models of homotopy types. For an exposition of the theory of simplicial sets in this spirit we suggest [6]. In modern view, simplicial sets are also used in parts of higher category theory that blend category theory and homotopy theory; expositions of this side of the theory of simplicial sets are in [7] and [8].

Definition 3 (Nerve)**.** Let X be a small category. The nerve of X, denoted $N(X)$, is a simplicial set defined by:

1. for any integer $n \geq 0$, $N(X)_n$ is the set of functors from $[n]$ to X,
2. for any arrow $[m] \xrightarrow{f} [l]$ in the simplex category, we define the value of $N(X)$ on f to be a function
$$N(X)(f) : N(X)_l \to N(X)_m$$
given by precomposition with f, so that we may write explicitly
$$N(X)(f)[g] = gf.$$

2.2 The classical (singular) cohomology of homotopy types presented as simplicial sets

Singular cohomology is an algebraic invariant of topological spaces. Since it is homotopy invariant, one can say that it is an invariant of homotopy types (indeed, singular cohomology, as a functor, factors through the localization of the category of topological spaces at homotopy equivalences). Homotopy types can, on the other hand, be presented by topological spaces and simplicial sets (among possibly other things, depending on what precisely is meant by „presented“). Therefore, singular cohomology should correspond to a construction on simplicial sets. It turns out that the appropriate construction for simplicial sets is already contained in the construction for topological spaces. So here, we in fact review a cohomological construction for simplicial sets, which yields singular cohomology of a topological space X, when the input simplicial set is the singular simplicial set of X. In order to make a distinction between the classical singular cohomology of topological spaces, and the cohomological construction presented here, we call our construction simplicial cohomology.

Let S be a simplicial set and A an abelian group. Denote by $S(\delta_j) = d_j : S_{n+1} \to S_n$, $0 \leq j \leq n+1$, the face maps. Define the chain complex associated to S to be:

$$0 \leftarrow \mathbb{Z}S_0 \xleftarrow{\partial_1} \mathbb{Z}S_1 \xleftarrow{\partial_2} \mathbb{Z}S_2 \xleftarrow{\partial_3} \mathbb{Z}S_3 \xleftarrow{\partial_4} \dots \tag{1}$$

where for $n \geq 0$, $\mathbb{Z}S_n$ is the free abelian group on the set S_n and for $n \geq 1$, $\partial_n = \sum_{j=0}^{n} (-1)^j d_j$.

We construct a cochain complex by applying the contravariant hom functor $\hom_{\mathrm{Ab}}(\bullet, A)$ to the diagram (1). More concisely, we define the cochain complex to be:

$$0 \to \hom_{\mathrm{Ab}}(\mathbb{Z}S_0, A) \xrightarrow{b_0} \hom_{\mathrm{Ab}}(\mathbb{Z}S_1, A) \xrightarrow{b_1} \hom_{\mathrm{Ab}}(\mathbb{Z}S_2, A) \xrightarrow{b_2} \dots$$

where for $n \geq 0$, $\hom_{\mathrm{Ab}}(\mathbb{Z}S_n, A)$ is the abelian group of group homomorphisms from $\mathbb{Z}S_n$ to A and $b_n = \hom_{\mathrm{Ab}}(\partial_{n+1}, A)$ is the precomposition by ∂_{n+1}.

The cohomology groups of S are defined as

$$H^0(S, A) = \ker(b_0) \text{ and for } n \geq 1,\ H^n(S, A) = \frac{\ker(b_n)}{\mathrm{im}(b_{n-1})}.$$

2.3 Cohomology of a small category

We have already described the nerve of a category and simplicial cohomology.

Definition 4. Cohomology of a small category X is the simplicial cohomology of the nerve $N(X)$.

3. Cohomology ring

In this section we describe a construction of a graded ring, constructed by equipping the cohomology groups with ring multiplication in a similar way as in singular cohomology. In fact, the construction of the multiplication operation presented here specializes to the multiplication operation on the singular cohomology of a topological space, when the simplicial set in question is the singular simplicial set of a topological space.[2]

Let A be a commutative ring. Denote, for each $n \geq 0$, by $C_n(X)$ the free abelian group on the set of functors $[n] \to X$. Also, let us denote by $C^n(X, A)$ the abelian group $\hom_{\mathrm{Ab}}(\mathbb{Z}N(X)_n, A)$ (note that we look at group homomorphisms into the underlying additive group of A) and accordingly by

$$b_n : C^n(X, A) \to C^{n+1}(X, A)$$

the accompanying maps in the cochain complex. Before moving on to the definition of the multiplication, we have to fix some notation. Restriction of functors to categories will be important later, especially along the following two types of morphisms in the simplex category.

Definition 5. For any integers $0 \leq a \leq b$ define a functor $l_{a,b} : [a] \to [b]$, $l_{a,b}(j) = j$ and $r_{ab} : [a] \to [b]$, $r_{a,b}(j) = b - a + j$.

Let $n \geq 0$ and $m \geq 0$ be nonnegative integers. The following equalities can be checked to hold through visualization:

$$\delta_j l_{n,n+m} = l_{n+1,n+m+1}\delta_j \text{ for } 0 \leq j \leq n+1, \tag{2}$$

$$\delta_{n+j} r_{m,n+m} = r_{m+1,n+m+1}\delta_j \text{ for } 0 \leq j \leq m+1. \tag{3}$$

Definition 6. For any integers $0 \leq n$, $0 \leq m$, define

$$C^n(X, A) \times C^m(X, A) \xrightarrow{\star_{nm}} C^{n+m}(X, A),$$

with $\alpha \star_{nm} \beta(\sigma) = \alpha(\sigma l_{n,n+m}) \cdot \beta(\sigma r_{m,n+m})$.

Lemma 7. For nonnegative integers n, m and $\alpha \in C^n(X, A)$, $\beta \in C^m(X, A)$, we have

$$b_{n+m}(\alpha \star_{nm} \beta) = b_n(\alpha) \star_{n+1,m} \beta + (-1)^n \alpha \star_{n,m+1} b_m(\beta).$$

Proof. Let $\sigma : [n + m + 1] \to X$ belong to $C_{n+m+1}(X)$. It suffices to show that $b_{n+m}(\alpha \star_{nm} \beta)(\sigma) = (b_n(\alpha) \star_{n+1,m} \beta)(\sigma) + (-1)^n(\alpha \star_{n,m+1} b_m(\beta))(\sigma)$. Observe what the first term on the right-hand side is equivalent to:

$$\begin{aligned}
(b_n(\alpha) \star_{n+1,m} \beta)(\sigma) &= b_n(\alpha)(\sigma l_{n+1,n+m+1})\beta(\sigma r_{m,n+m+1}) \\
&= \alpha(\partial_{n+1}(\sigma l_{n+1,n+m+1}))\beta(\sigma r_{m,n+m+1}) \\
&= \alpha(\sum_{j=0}^{n+1}(-1)^j \sigma l_{n+1,n+m+1}\delta_j)\beta(\sigma r_{m,n+m+1}) \\
&= \sum_{j=0}^{n+1}(-1)^j \alpha(\sigma l_{n+1,n+m+1}\delta_j)\beta(\sigma r_{m,n+m+1}) \\
&= \sum_{j=0}^{n+1}(-1)^j \alpha(\sigma \delta_j l_{n,n+m})\beta(\sigma r_{m,n+m+1}).
\end{aligned}$$

[2] The singular simplicial set of a topological space X is sometimes also called the singular complex of X.

Notice in the last line we used (2). Similarly for the second term on the right-hand side:

$$\begin{aligned}(-1)^n(\alpha \star_{n,m+1} b_m(\beta))(\sigma) &= (-1)^n\alpha(\sigma l_{n,n+m+1})\beta(\partial_{m+1}(\sigma r_{m+1,n+m+1})) \\ &= (-1)^n\alpha(\sigma l_{n,n+m+1})\beta(\sum_{j=0}^{m+1}(-1)^j\sigma r_{m+1,n+m+1}\delta_j) \\ &= \sum_{j=n}^{n+m+1}(-1)^j\alpha(\sigma l_{n,n+m+1})\beta(\sigma\delta_j r_{m,n+m}).\end{aligned}$$

In the last line we used (3) and then changed variables. The last term of $(b_n(\alpha) \star_{n+1,m} \beta)(\sigma)$ and the first term of $(-1)^n(\alpha \star_{n,m+1} b_m(\beta))(\sigma)$ cancel each other. The remaining terms are precisely $b_{n+m}(\alpha \star_{nm} \beta)(\sigma)$. ∎

Lemma 8. We have the following statements:

1. if $\alpha \in \ker(b_n)$ and $\beta \in \ker(b_m)$, then $\alpha \star_{nm} \beta \in \ker(b_{n+m})$,
2. if $\alpha \in \operatorname{im}(b_{n-1})$ and $\beta \in \ker(b_m)$, then $\alpha \star_{nm} \beta \in \operatorname{im}(b_{n+m-1})$,
3. if $\alpha \in \ker(b_n)$ and $\beta \in \operatorname{im}(b_{m-1})$, then $\alpha \star_{nm} \beta \in \operatorname{im}(b_{n+m-1})$,
4. $\star_{nm}$ is bilinear, in other words, it distributes over addition on both sides.

Proof. We check each of the statements individually.

1. Assume $\alpha \in \ker(b_n)$ and $\beta \in \ker(b_m)$. Then $b_{n+m}(\alpha \star_{nm} \beta) = b_n(\alpha) \star_{n+1,m} \beta + (-1)^n\alpha \star_{n,m+1} b_m(\beta) = 0 + 0 = 0$.
2. Assume $\alpha = b_{n-1}(\zeta)$ and $\beta \in \ker(b_m)$. Then $b_{n+m-1}(\zeta \star_{n-1,m} \beta) = b_{n-1}(\zeta) \star_{nm} \beta + (-1)^{n-1}\zeta \star_{n-1,m+1} b_m(\beta) = \alpha \star_{nm} \beta + 0 = \alpha \star_{nm} \beta$.
3. Assume $\alpha \in \ker(b_n)$ and $\beta = b_{m-1}(\zeta)$. Then $b_{n+m-1}(\alpha \star_{n,m-1} \zeta) = b_n(\alpha) \star_{n+1,m-1} \zeta + (-1)^n\alpha\star_{nm}b_{m-1}(\zeta) = 0+(-1)^n\alpha\star_{nm}\beta = (-1)^n\alpha\star_{nm}\beta$. Then $\alpha\star_{nm}\beta = (-1)^n b_{n+m-1}(\alpha\star_{n,m-1} \zeta) = b_{n+m-1}((-1)^n\alpha \star_{n,m-1} \zeta)$.
4. Let $\alpha, \beta \in C^n(X, A)$ and $\zeta, \omega \in C^m(X, A)$. Let $\sigma : [n + m] \to X$ be arbitrary. Then

$$\begin{aligned}((\alpha + \beta) \star_{nm} \zeta)(\sigma) &= (\alpha + \beta)(\sigma l_{n,n+m})\zeta(\sigma r_{m,n+m}) \\ &= (\alpha(\sigma l_{n,n+m}) + \beta(\sigma l_{n,n+m}))\zeta(\sigma r_{m,n+m}) \\ &= \alpha(\sigma l_{n,n+m})\zeta(\sigma r_{m,n+m}) + \beta(\sigma l_{n,n+m})\zeta(\sigma r_{m,n+m}) \\ &= (\alpha \star_{nm} \zeta)(\sigma) + (\beta \star_{nm} \zeta)(\sigma) \\ &= (\alpha \star_{nm} \zeta + \beta \star_{nm} \zeta)(\sigma)\end{aligned}$$

   which shows that $(\alpha + \beta) \star_{nm} \zeta = \alpha \star_{nm} \zeta + \beta \star_{nm} \zeta$. A similar calculation shows that $\alpha \star_{nm} (\zeta + \omega) = \alpha \star_{nm} \zeta + \alpha \star_{nm} \omega$. ∎

Lemma 8 implies there is a well defined map

$$H^n(X, A) \times H^m(X, A) \xrightarrow{\smile_{nm}} H^{n+m}(X, A)$$

$$(\alpha + \operatorname{im}(b_{n-1}), \beta + \operatorname{im}(b_{m-1})) \longmapsto \alpha \star_{nm} \beta + \operatorname{im}(b_{n+m-1}),$$

which distributes over addition on both sides.

Definition 9. Let X be a small category. Then define:

1. The cohomology of X with coefficients in A, as an abelian group, is:
$$H^*(X,A) = \bigoplus_{n=0}^{\infty} H^n(X,A).$$
2. The cup product is the map $H^*(X,A) \times H^*(X,A) \xrightarrow{\smile} H^*(X,A)$, prescribed with:
$$(\bigoplus_{n\geq 0} \omega_n) \smile (\bigoplus_{m\geq 0} \varphi_m) = \bigoplus_{k\geq 0} (\sum_{j=0}^{k} \omega_j \smile_{j,k-j} \varphi_{k-j}).$$

Theorem 10. $H^*(X,A)$ with addition and the cup product is a ring.

Proof. We have to show that $\smile$ makes $H^*(X,A)$ into a monoid and that $\smile$ distributes over addition.

In this paragraph we prove associativity of $\smile$. The $\star$ operation is associative in the sense that for any nonnegative integers n, m and l, $\star_{n+m,l} \circ (\star_{nm} \times 1_{C^l(X,A)}) = \star_{n,m+l} \circ (1_{C^n(X,A)} \times \star_{m,l})$. Therefore
$$\smile_{n+m,l} \circ (\smile_{nm} \times 1_{H^l(X,A)}) = \smile_{n,m+l} \circ (1_{H^n(X,A)} \times \smile_{m,l}).$$
Let $\omega = \bigoplus_{n\geq 0} \omega_n$, $\varphi = \bigoplus_{m\geq 0} \varphi_m$ and $\zeta = \bigoplus_{l\geq 0} \zeta_l$ be elements of $H^*(X,A)$. Then we have:
$$\begin{aligned}
(\omega \smile \varphi) \smile \zeta &= (\bigoplus_{m\geq 0} (\sum_{j=0}^{m} \omega_j \smile_{j,m-j} \varphi_{m-j})) \smile (\bigoplus_{k\geq 0} \zeta_k) \\
&= \bigoplus_{m\geq 0} (\sum_{l=0}^{m} (\sum_{j=0}^{l} \omega_j \smile_{j,l-j} \varphi_{l-j}) \smile_{l,m-l} \zeta_{m-l}) \\
&= \bigoplus_{m\geq 0} (\sum_{l=0}^{m} \sum_{j=0}^{l} (\omega_j \smile_{j,l-j} \varphi_{l-j}) \smile_{l,m-l} \zeta_{m-l}) \\
&= \bigoplus_{m\geq 0} (\sum_{l=0}^{m} \sum_{j=0}^{l} \omega_j \smile_{j,m-j} (\varphi_{l-j} \smile_{l-j,m-l} \zeta_{m-l})) \\
&= \bigoplus_{m\geq 0} (\sum_{j=0}^{m} \sum_{l=j}^{m} \omega_j \smile_{j,m-j} (\varphi_{l-j} \smile_{l-j,m-l} \zeta_{m-l})) \\
&= \bigoplus_{m\geq 0} (\sum_{j=0}^{m} \omega_j \smile_{j,m-j} (\sum_{l=j}^{m} \varphi_{l-j} \smile_{l-j,m-l} \zeta_{m-l})) \\
&= \bigoplus_{m\geq 0} (\sum_{j=0}^{m} \omega_j \smile_{j,m-j} (\sum_{l=0}^{m-j} \varphi_{l} \smile_{l,m-j-l} \zeta_{m-j-l})) \\
&= \bigoplus_{m\geq 0} (\sum_{j=0}^{m} \omega_j \smile_{j,m-j} (\varphi \smile \zeta)_{m-j}) \\
&= \omega \smile (\varphi \smile \zeta).
\end{aligned}$$

In this paragraph we describe the neutral element. Define an element $\tilde{e}$ of $H^0(X,A) = \ker(b_0)$ with $\tilde{e}(x) = 1 \in A$ for any object $x \in X$. A quick calculation shows that for any $\alpha \in C^n(X,A)$, $n \geq 0$, $\tilde{e} \star_{0n} \alpha = \alpha \star_{n0} \tilde{e} = \alpha$. Define $e \in H^*(X,A)$ with $(e_j)_{j\geq 0} = (\tilde{e}, 0, 0, \dots)$. For any $\omega \in H^*(X,A)$,
$$e \smile \omega = \bigoplus_{j\geq 0} (\tilde{e} \smile_{0,j} \omega_j) = \omega$$

and similarly $\omega \smile e = \omega$.

We now turn to distributivity. Let once again $\omega, \varphi, \zeta \in H^*(X, A)$. Then:

$$\begin{aligned}(\omega+\varphi) \smile \zeta &= \bigoplus_{m\geq 0}(\sum_{j=0}^{m}(\omega_j+\varphi_j) \smile_{j,m-j} \zeta_{m-j}) \\ &= \bigoplus_{m\geq 0}(\sum_{j=0}^{m}(\omega_j \smile_{j,m-j} \zeta_{m-j} + \varphi_j \smile_{j,m-j} \zeta_{m-j}) \\ &= \omega \smile \zeta + \varphi \smile \zeta.\end{aligned}$$

A similar calculation shows that $\omega \smile (\varphi+\zeta) = \omega \smile \varphi + \omega \smile \zeta$. ∎

4. Functoriality and homotopy invariance

In this section we review a notion of homotopy in Cat, giving rise to a homotopy category hCat. Then we show that $H^*(\bullet, A)$ is a homotopy invariant cofunctor $\mathrm{Cat}^{\mathrm{op}} \to \mathrm{Ring}$, as displayed by the following diagram

$$\begin{tikzcd} \mathrm{Cat}^{\mathrm{op}} \arrow[r] \arrow[d] & \mathrm{Ring} \\ (\mathrm{hCat})^{\mathrm{op}}. \arrow[ur] & \end{tikzcd}$$

Let us begin our description of homotopy between functors.

Definition 11. $\mathfrak{Z}$ is the category with objects $\mathrm{Ob}(\mathfrak{Z}) = \mathbb{Z}$ the integers, and the morphism sets have the following description. If $n \in \mathbb{Z}$ is an even integer, i.e. $n = 2k$ for some $k \in \mathbb{Z}$, then $\mathfrak{Z}(n, n+1)$ is a singleton and $\mathfrak{Z}(n+1, n) = \emptyset$. If $n \in \mathbb{Z}$ is odd, i.e. $n = 2k+1$ for some $k \in \mathbb{Z}$, then $\mathfrak{Z}(n, n+1) = \emptyset$ and $\mathfrak{Z}(n+1, n)$ is a singleton. If $n \neq m$ and $|m-n| \geq 2$, then there are no morphisms between n and m. Each integer n admits a unique endomorphism – its identity endomorphism. Composition is defined in a natural way (note that each composable pair of arrows contains an identity).

Definition 12. Suppose n and m are integers, $n \leq m$. We denote by W_{nm} the full subcategory of $\mathfrak{Z}$ spanned by the set of integers $\{k; n \leq k \leq m\}$.

Definition 13. Let $\mathcal{C}$ be a category. Objects x and y of $\mathcal{C}$ are connected, if there exist integers $n \leq m$ and a functor $W_{nm} \xrightarrow{\gamma} \mathcal{C}$ with $\gamma(n) = x$ and $\gamma(m) = y$.

Definition 14. Two functors $F : A \to B$ and $G : A \to B$, where A and B are small categories are called homotopic, if they are connected in the functor category $\mathrm{Fun}(A, B)$. The related functor $H : W_{nm} \to \mathrm{Fun}(A, B)$ with $H(n) = F$ and $H(m) = G$ is called a homotopy from F to G and we write $H : F \simeq G$.

Remark 15. Equivalently, a homotopy between functors is a finite zig-zag of natural transformations between them. Some justification for this notion of homotopy between functors seems to be appropriate. Recall that simplicial sets were devised as combinatorial models for homotopy types. More formally, the category of simplicial sets has a model category structure, called the standard model category of simplicial sets (see [9]). If $f, g : X \to Y$ are simplicial maps between simplicial sets X and Y, then a left homotopy from f to g is a simplicial map $\theta : X \times \Delta^1 \to Y$ such that $\theta|_{X\times\{0\}} = f$ and $\theta|_{X\times\{1\}} = g$. This is in analogy with the definition of homotopy $X \times [0,1] \to Y$

between continuous maps $X \to Y$ of topological spaces. Note that indeed the two definitions differ mainly in replacing the interval $[0,1]$ with the 1-simplex Δ^1. Each natural transformation

$$\eta : X \times [1] \to Y,$$

where X and Y are small categories, induces a homotopy

$$N(\eta) : N(X \times [1]) \cong N(X) \times \Delta^1 \to N(Y)$$

via the nerve functor.[3] Any two homotopic functors in our sense determine homotopic maps of simplicial sets.

Definition 16. Let $\mathcal{C}$ be a category. A congruence on $\mathcal{C}$ is a specification of an equivalence relation $\sim$ on each hom set, such that for any $f, g \in \mathcal{C}(X,Y)$ and any $h, p \in \mathcal{C}(Y,Z)$, if $f \sim g$ and $h \sim p$, then $pf \sim hg$. Arrows $f \sim g$ are called congruent.

Lemma 17, below, asserts that any congruence in a category gives rise to another category, where congruent arrows are identified.

Lemma 17. If $\mathcal{C}$ is a category with a congruence $\sim$, we can construct a new category which we denote $\mathcal{C}_{/\sim}$. Objects of $\mathcal{C}_{/\sim}$ are objects of $\mathcal{C}$, $Ob(\mathcal{C}_{/\sim}) = Ob(\mathcal{C})$, hom sets of $\mathcal{C}_{/\sim}$ are quotient sets of hom sets of $\mathcal{C}$ determined by the equivalence relation given by $\sim$,

$$\forall X, Y \in \mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathcal{C}_{/\sim})\colon\ \mathcal{C}_{/\sim}(X,Y) = \mathcal{C}(X,Y)_{/\sim}$$

and composition in $\mathcal{C}_{/\sim}$ descends from composition in $\mathcal{C}$, in the sense that for arbitrary objects $X, Y, Z \in \mathcal{C}_{/\sim}$ we have a commutative square

$$\begin{tikzcd}
\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \arrow[r] \arrow[d] & \mathcal{C}(X,Z) \arrow[d] \\
\mathcal{C}_{/\sim}(X,Y) \times \mathcal{C}_{/\sim}(Y,Z) \arrow[r] & \mathcal{C}_{/\sim}(X,Z),
\end{tikzcd}$$

where the vertical arrows are the canonical maps $f \mapsto [f]$. There is a functor $P : \mathcal{C} \to \mathcal{C}_{/\sim}$, sending $X \mapsto X$, $(X \xrightarrow{f} Y) \mapsto [f]$, where $[f]$ is the equivalence class of f in $\mathcal{C}(X,Y)_{/\sim}$.

For any functor $F : \mathcal{C} \to \mathcal{D}$ with the property that $f \sim g$ implies $F(f) = F(g)$, there exists a unique functor $\tilde{F} : \mathcal{C}_{/\sim} \to \mathcal{D}$ with $\tilde{F} \circ P = F$.

Proof. We define $\mathcal{C}_{/\sim}$ as the category with the following description. The set of objects of $\mathcal{C}_{/\sim}$ is equal to the set of objects of $\mathcal{C}$. For each pair of objects X and Y in $\mathcal{C}_{/\sim}$ (or, equivalently, in $\mathcal{C}$), we define $\hom_{\mathcal{C}_{/\sim}}(X,Y)$ to be the set of equivalence classes of the equivalence relation $\sim$ on $\hom_{\mathcal{C}}(X,Y)$. For each morphism $f : X \to Y$ in $\mathcal{C}$, let us denote its equivalence class by $[f]$. Because $\sim$ is a congruence, we have that we can define composition in $\mathcal{C}_{/\sim}$ by letting $[g][f] = [gf]$ for each pair of equivalence classes of morphisms $[g] : Y \to Z$ and $[f] : X \to Y$. One can check that $\mathcal{C}_{/\sim}$ is a category. Furthermore, we can define a functor $P : \mathcal{C} \to \mathcal{C}_{/\sim}$ with the following description. For each object X in $\mathcal{C}$, we have $P(X) = X$. For each morphism $f : X \to Y$ in $\mathcal{C}$, we have $P(f) = [f]$.

To check the universal property of P, we first note that $f \sim g$ implies $P(f) = P(g)$ by construction. Suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor such that $f \sim g$ implies $F(f) = F(g)$. We wish to show there exists a unique functor $\tilde{F} : \mathcal{C}_{/\sim} \to \mathcal{D}$ such that $\tilde{F}P = F$. For uniqueness, suppose we have two functors, $A, B : \mathcal{C}_{/\sim} \to \mathcal{D}$ such that $AP = BP = F$. Then for each object X in $\mathcal{C}_{/\sim}$ we have

[3] The nerve functor N is a right adjoint, and thus commutes with products. Also, there is an equality $N([1]) = \Delta^1$ of simplicial sets (not just an isomorphism).

$A(X) = AP(X) = BP(X) = B(X)$. This means A and B have equal values on objects. Also, for each morphism $[f] : X \to Y$ in $\mathcal{C}_{/\sim}$ we have $A([f]) = AP(f) = BP(f) = B([f])$. This means A and B have equal values on morphisms. Therefore, A and B are equal. To show existence, define a functor $\tilde{F} : \mathcal{C}_{/\sim} \to \mathcal{D}$ as follows. For each object X in $\mathcal{C}_{/\sim}$, we let $\tilde{F}(X) = F(X)$. For each morphism $[f] : X \to Y$, we let $\tilde{F}([f]) = F(f)$. We assumed $f \sim g$ implies $F(f) = F(g)$, so $\tilde{F}$ is well defined on morphisms. One can check that $\tilde{F}$ is a functor, and the equality $\tilde{F}P = F$ holds by construction. ∎

Lemma 18. Homotopy $\simeq$ is a congruence in the category of small categories Cat, and therefore gives rise to another category $\mathrm{Cat}_{/\simeq} = \mathrm{hCat}$.

Proof. In this paragraph we show that for any two small categories A and B, homotopy $\simeq$ is an equivalence relation on the set of functors $\mathrm{Cat}(A, B)$.

1. Reflexivity. If $F : A \to B$ is a functor, then W_{nn} (for any integer n) is the category with one object and only the identity morphism. A functor $W_{nn} \to \mathrm{Fun}(A, B)$ is equivalent to a choice of object of $\mathrm{Fun}(A, B)$, i.e. a functor $A \to B$. Choosing $F : A \to B$ gives a functor $W_{nn} \to \mathrm{Fun}(A, B)$ which is a homotopy $F \simeq F$.

2. Symmetry. Let $\theta : W_{nm} \to \mathrm{Fun}(A, B)$ be a homotopy $\theta : F \simeq G$. Precomposing θ with a functor $W_{-m,-n} \to W_{nm}$, $j \mapsto -j$ gives a functor $\theta^- : W_{-m,-n} \to \mathrm{Fun}(A, B)$ with $\theta^-(-m) = G$ and $\theta^-(-n) = F$.

3. Transitivity. Let $\theta : W_{nm} \to \mathrm{Fun}(A, B)$, $\theta : F \simeq G$ and $\gamma : W_{lr} \to \mathrm{Fun}(A, B)$, $\gamma : G \simeq K$. If $(-1)^m = (-1)^l$, then define $W_{n,r+m-l} \to \mathrm{Fun}(A, B)$ by concatenating θ and γ and this is a homotopy $F \simeq K$. If $(-1)^m = -(-1)^l$, then define $W_{n,r+m-l-1} \to \mathrm{Fun}(A, B)$ by joining the zig-zags of natural transformations and composing the natural transformations to and from G.

So we have proven that $\simeq$ is an equivalence relation on hom sets of Cat.

Suppose $\theta : f \simeq g$ are homotopic functors between small categories $X \to Y$, ($\theta : W_{nm} \to \mathrm{Fun}(X, Y)$) and $\gamma : h \simeq p$ are homotopic functors between small categories $Y \to Z$, ($\gamma : W_{lr} \to \mathrm{Fun}(Y, Z)$). Then define $W_{nm} \to \mathrm{Fun}(X, Z)$, $j \mapsto h\theta(j)$ and this is a homotopy $hf \simeq hg$. Then define $W_{lr} \to \mathrm{Fun}(X, Z)$, $j \mapsto \gamma(j)g$ and this is a homotopy $hg \simeq pg$. Transitivity implies $hf \simeq pg$. ∎

Let us review why $H^*(\bullet, A)$ is cofunctorial. If $X \xrightarrow{f} Y$ is a functor between small categories, then there are natural maps (group homomorphisms) $C^n(f, A) : C^n(Y, A) \to C^n(X, A)$ acting as a pullback along the natural map $\mathbb{Z}N(f)_n : \mathbb{Z}N(X)_n \to \mathbb{Z}N(Y)_n$ (also group homomorphisms), for any integer $n \geq 0$. This collection of maps is a morphism of cochain complexes of abelian groups and therefore descends onto homomorphisms of cohomology groups; we get a collection of maps $H^n(f, A) : H^n(Y, A) \to H^n(X, A)$ for $n \geq 0$. Forming a direct sum of these gives what we denote $f^* = H^*(f, A) : H^*(Y, A) \to H^*(X, A)$.

Lemma 19. If $X \xrightarrow{f} Y$ is a functor between small categories, then for any integer $n \geq 0$:

1. $H^n(f, A) : H^n(Y, A) \to H^n(X, A)$ is additive,

2. if $0 \leq j \leq n$, $\omega_j \in H^j(Y, A)$ and $\varphi_{n-j} \in H^{n-j}(Y, A)$, then

$$H^n(f, A)(\omega_j \smile_{j,n-j} \varphi_{n-j}) = H^j(f, A)(\omega_j) \smile_{j,n-j} H^{n-j}(f, A)(\varphi_{n-j}).$$

Proof. Let $n \geq 0$ be any nonnegative integer.

1. Let $\alpha, \beta \in H^n(Y, A)$, then

$$\begin{aligned} H^n(f,A)(\alpha+\beta) &= H^n(f,A)([\tilde{\alpha}] + \left[\tilde{\beta}\right]) \\ &= H^n(f,A)(\left[\tilde{\alpha} + \tilde{\beta}\right]) \\ &= \left[(\tilde{\alpha} + \tilde{\beta}) \circ \mathbb{Z}N(f)_n\right] \\ &= \left[\tilde{\alpha} \circ \mathbb{Z}N(f)_n + \tilde{\beta} \circ \mathbb{Z}N(f)_n\right] \\ &= [\tilde{\alpha} \circ \mathbb{Z}N(f)_n] + \left[\tilde{\beta} \circ \mathbb{Z}N(f)_n\right] \\ &= H^n(f,A)(\alpha) + H^n(f,A)(\beta). \end{aligned}$$

2. Let ω_j and φ_{n-j} be as in the lemma. Then

$$\begin{aligned} H^n(f,A)(\omega_j \smile_{j,n-j} \varphi_{n-j}) &= H^n(f,A)([\tilde{\omega}] \smile_{j,n-j} [\tilde{\varphi}]) \\ &= H^n(f,A)([\tilde{\omega} \star_{j,n-j} \tilde{\varphi}]) \\ &= [(\tilde{\omega} \star_{j,n-j} \tilde{\varphi}) \circ \mathbb{Z}N(f)_n] \\ &= [(\tilde{\omega} \circ \mathbb{Z}N(f)_j) \star_{j,n-j} (\tilde{\varphi} \circ \mathbb{Z}N(f)_{n-j})] \\ &= H^j(f,A)(\omega_j) \smile_{j,n-j} H^{n-j}(f,A)(\varphi_{n-j}). \end{aligned}$$

∎

Lemma 20. If $X \xrightarrow{f} Y$ is a functor between small categories, then $f^* : H^*(Y, A) \to H^*(X, A)$ is a ring homomorphism.

Proof. We have to prove that f^* is additive, multiplicative and preserves unit.

1. Additivity. Let $\omega, \varphi \in H^*(Y, A)$. Then

$$\begin{aligned} f^*(\omega+\varphi) &= f^*(\bigoplus_{n\geq 0}(\omega_n + \varphi_n)) \\ &= \bigoplus_{n\geq 0} H^n(f,A)(\omega_n+\varphi_n) \\ &= \bigoplus_{n\geq 0}(H^n(f,A)(\omega_n) + H^n(f,A)(\varphi_n)) \\ &= (\bigoplus_{n\geq 0} H^n(f,A)(\omega_n)) + (\bigoplus_{n\geq 0} H^n(f,A)(\varphi_n)) \\ &= H^*(f,A)(\omega) + H^*(f,A)(\varphi). \end{aligned}$$

2. Multiplicativity. Let $\omega, \varphi \in H^*(Y, A)$. Then

$$\begin{aligned} f^*(\omega \smile \varphi) =& f^*(\bigoplus_{n\geq 0}\sum_{j=0}^{n} \omega_j \smile_{j,n-j} \varphi_{n-j}) \\ &= \bigoplus_{n\geq 0} H^n(f,A)(\sum_{j=0}^{n} \omega_j \smile_{j,n-j} \varphi_{n-j}) \\ &= \bigoplus_{n\geq 0}\sum_{j=0}^{n} H^n(f,A)(\omega_j \smile_{j,n-j} \varphi_{n-j}) \\ &= \bigoplus_{n\geq 0}\sum_{j=0}^{n} H^j(f,A)(\omega_j) \smile_{j,n-j} H^{n-j}(f,A)(\varphi_{n-j}) \\ &= H^*(f,A)(\omega) \smile H^*(f,A)(\varphi). \end{aligned}$$

3. Similarly $H^*(f, A)(1) = 1$. ■

Theorem 21. The construction $H^*(\bullet, A)$ is contravariantly functorial. In other words, we have a cofunctor

$$H^*(\bullet, A) : \mathrm{Cat}^{\mathrm{op}} \to \mathrm{Ring},$$

from the category of small categories to the category of rings.

Proof. If $\mathrm{id}_X : X \to X$ is the identity functor, we see that $H^*(\mathrm{id}_X, A) = \mathrm{id}_{H^*(X,A)}$.

If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a composable pair of arrows, we want to show that the following diagram with induced ring homomorphisms commutes:

$$\begin{tikzcd} H^*(Z,A) \arrow[r] \arrow[dr] & H^*(Y,A) \arrow[d] \\ & H^*(X,A). \end{tikzcd}$$

It is easy to see that the following two diagrams with the obvious maps commute for all integers $n \geq 0$:

$$\begin{tikzcd} \mathbb{Z}N(X)_n \arrow[r] \arrow[dr] & \mathbb{Z}N(Y)_n \arrow[d] \\ & \mathbb{Z}N(Z)_n, \end{tikzcd} \qquad \begin{tikzcd} C^n(Z,A) \arrow[r] \arrow[dr] & C^n(Y,A) \arrow[d] \\ & C^n(X,A). \end{tikzcd}$$

In fact, the right-hand-side triangle above results from the application of $\hom_{\mathrm{Ab}}(\bullet, A)$ to the left-hand-side triangle above. So for all integers $n \geq 0$ we have

$$\begin{tikzcd} H^n(Z,A) \arrow[r] \arrow[dr] & H^n(Y,A) \arrow[d] \\ & H^n(X,A). \end{tikzcd}$$

Forming a direct sum implies $(gf)^* = f^*g^*$. ■

We now come closer to homotopy invariance of $H^*(\bullet, A)$.

Lemma 22. Let $\mu : f \to g$ be a natural transformation between functors $f,\ g : X \to Y$. Then $H^*(f, A) = H^*(g, A)$.

Proof. $H^*(f, A) = H^*(g, A)$ if and only if $\forall n \geq 0$ $H^n(f, A) = H^n(g, A)$, so this second property is what we aim to prove for all integers $n \geq 0$. We prove it in steps:

1. natural transformation $f \xrightarrow{\mu} g$ leads to a canonical construction of a homotopy between morphisms of chain complexes $\{\mathbb{Z}N(f)_n : \mathbb{Z}N(X)_n \to \mathbb{Z}N(Y)_n, n \geq 0\}$ and $\{\mathbb{Z}N(g)_n : \mathbb{Z}N(X)_n \to \mathbb{Z}N(Y)_n, n \geq 0\}$.

2. Recall that if $F : \mathrm{Ab}^{\mathrm{op}} \to \mathrm{Ab}$ is an additive cofunctor from the category of abelian groups to itself, and L and E are chain complexes in the category of abelian groups, $u, v : L \to E$ morphisms of chain complexes, and $\theta : u \simeq v$ a chain homotopy, then F induces a homotopy $F(\theta) : F(u) \simeq F(v)$, where $F(u) : F(L) \to F(E)$ and $F(v) : F(L) \to F(E)$ are the induced maps of chain complexes. In particular, when we take $F = \hom_{\mathrm{Ab}}(\bullet, A)$, we get a chain homotopy between maps of cochain complexes

$$\{C^n(f, A) : C^n(Y, A) \to C^n(X, A), n \geq 0\}$$

and

$$\{C^n(g,A) : C^n(Y,A) \to C^n(X,A), n \geq 0\}.$$

3. Recall that if $u \simeq v : L \to E$ are homotopic morphisms of cochain complexes, then $H^n(u) = H^n(v)$ for all $n \geq 0$. In particular, by the previous point $H^n(f,A) = H^n(g,A)$ for all $n \geq 0$.

To complete the proof, we thus only have to work out point 1 from this list.

We want to define a homotopy s from $\{\mathbb{Z}N(f)_n : \mathbb{Z}N(X)_n \to \mathbb{Z}N(Y)_n, n \geq 0\}$ to $\{\mathbb{Z}N(g)_n : \mathbb{Z}N(X)_n \to \mathbb{Z}N(Y)_n, n \geq 0\}$, i.e. a set of group homomorphisms $s_n : \mathbb{Z}N(X)_n \to \mathbb{Z}N(Y)_{n+1}$, $n \geq 0$ such that for $n \geq 1$, $\partial^Y_{n+1} s_n + s_{n-1}\partial^X_n = \mathbb{Z}N(g)_n - \mathbb{Z}N(f)_n$ and $\partial^Y_1 s_0 = \mathbb{Z}N(g)_0 - \mathbb{Z}N(f)_0$. It is enough to define s_n on generators $([n] \xrightarrow{\sigma} X) \in N(X)_n \subseteq \mathbb{Z}N(X)_n$ and extend s_n by additivity. Let's set about defining what $s_n(\sigma)$ should be.

A natural transformation $\mu : f \to g$ is equivalently a functor (which we will denote by the same letter) $\mu : X \times [1] \to Y$ with the following diagram commuting:

$$\begin{tikzcd} X \cong X \times [0] \arrow[r] \arrow[dr] & X \times [1] \arrow[d] & X \times [0] = X \arrow[l] \arrow[dl] \\ & Y. & \end{tikzcd}$$

Denote $\vartheta_\sigma : [n] \times [1] \to Y$, $\vartheta_\sigma = \mu \circ (\sigma \times \mathrm{id}_{[1]})$. Denote $f_{j,n} : [n+1] \to [n] \times [1]$, $0 \leq j \leq n$ functors with

$$f_{j,n}(i) = \begin{cases} (i,0), & \text{if } 0 \leq i \leq j \\ (i-1,1), & \text{if } j+1 \leq i \leq n+1. \end{cases}$$

Define $s_n(\sigma) = \sum_{j=0}^{n} (-1)^j \vartheta_\sigma \circ f_{j,n}$. Seeing that $(s_n; n \geq 0)$ defines a homotopy is done through direct calculation, which we leave as an exercise. ■

Theorem 23. If $X \xrightarrow{f} Y$, $X \xrightarrow{g} Y$ are homotopic functors between small categories, $f \backsimeq g$, then $H^*(f,A) = H^*(g,A)$.

Proof. By assumption there exists a homotopy $\theta : W_{nm} \to \mathrm{Fun}(X,Y)$, $\theta : f \backsimeq g$. Then $H^*(f,A) = H^*(\theta(n+1),A) = \cdots = H^*(\theta(m-1),A) = H^*(g,A)$, where we use Lemma 22 in each equality. ■

5. Cup length and Lusternik–Schnirelmann category

The Lusternik–Schnirelmann category is a nonnegative integer, traditionally associated to a topological space. In the last couple of years, the Lusternik–Schnirelmann category has been generalized to categories by K. Tanaka. A classical theorem says that the cup-length of a topological space is a lower bound on the Lusternik–Schnirelmann category of that space. In this section we prove an analogous result for small categories, with the appropriately modified notions of cup-length and Lusternik–Schnirelmann category, so that they apply to small categories.

From here on, a category is assumed to be **connected**, meaning that any two objects are connected. If X is a (connected) category, then any two functors $a, b : [0] \to X$ are homotopic. By Theorem 23, they give the same ring homomorphism $H^*(X,A) \to H^*([0],A)$. The kernel of this ring homomorphism is an ideal and is called the **reduced cohomology** of X, denoted $\tilde{H}^*(X,A)$.

Definition 24. Let X be a small category. Then the cup length of X is

$$\mathrm{cl}(X) = \max\{n \geq 0 \mid \exists \omega_1, \ldots, \omega_n \in \tilde{H}^*(X,A) \text{ with } \omega_1 \smile \cdots \smile \omega_n \neq 0\}.$$

In words, the cup-length of X is the greatest nonnegative integer n for which there exist elements $\omega_1, \ldots, \omega_n \in \tilde{H}^*(X,A)$ with nonzero product; $\omega_1 \smile \cdots \smile \omega_n \neq 0$.

The next few definitions serve the purpose of defining the Lusternik–Schnirelmann category (LS-category) of a small category.

Definition 25. A subcategory U of X is contractible in X, if the inclusion functor $U \hookrightarrow X$ is homotopic to some constant functor.

If U is a subcategory of X, contractible in X, then the inclusion functor $\iota : U \hookrightarrow X$, homotopic to a constant functor $c : U \to X$, gives $H^*(\iota, A) = H^*(c, A) : H^*(X, A) \to H^*(U, A)$.

Definition 26. A geometric cover of a category X is a set of subcategories $\{V_j, j \in J\}$ such that any finite sequence of composable arrows $f_1, \ldots, f_n$ is in V_j for some $j \in J$.

Definition 27. The LS-category $\operatorname{cat}(X)$ is the smallest integer $n \geq 1$ such that there exists a finite geometric cover $\{V_j\}_{j=1}^n$ with V_j contractible in X for all $j = 1, \ldots, n$. If no such cover exists, we put $\operatorname{cat}(X) = \infty$.

Let $U \xrightarrow{\iota} X$ be the inclusion functor of a subcategory inside a small category X. We can naturally form a short exact sequence of chains,

$$0 \longrightarrow C_n(U) \longrightarrow C_n(X) \longrightarrow C_n(X)/C_n(U) \longrightarrow 0$$

for every nonnegative integer $n \geq 0$. Instead of viewing the above short exact sequence in each degree, we can view it as a short exact sequence of chain complexes:

$$0 \longrightarrow C_\bullet(U) \longrightarrow C_\bullet(X) \longrightarrow C_\bullet(X)/C_\bullet(U) \longrightarrow 0.$$

When applying the hom cofunctor $\hom_{\mathrm{Ab}}(\bullet, A)$, the arrows reverse direction and it is part of our next task to show that the resulting short sequence is again exact. We denote $C^\bullet(X, U, A) = Hom_{\mathrm{Ab}}(\frac{C_\bullet(X)}{C_\bullet(U)}, A)$.

Lemma 28. The sequence

$$0 \to C^\bullet(X, U, A) \xrightarrow{p^*} C^\bullet(X, A) \xrightarrow{C^\bullet(\iota)} C^\bullet(U, A) \to 0$$

is a short exact sequence.

Proof. The sequence is exact if and only if for all $n \geq 0$

$$0 \to C^n(X, U, A) \xrightarrow{p_n^*} C^n(X, A) \xrightarrow{C^n(\iota)} C^n(U, A) \to 0$$

is a short exact sequence. Proving this is a list of verifications:

1. The map $C^n(X, U, A) \xrightarrow{p_n^*} C^n(X, A)$ is *injective.* Let $\alpha \in C^n(X, U, A)$ be nonzero, meaning there exists an element $x \in \frac{C_n(X)}{C_n(U)}$ with $\alpha(x) \neq 0$. Then $p_n^*(\alpha)_{|p_n^{-1}(x)} = (\alpha p_n)_{|p_n^{-1}(x)} \neq 0$. So $\ker(p_n^*)$ is trivial.

2. We have inclusion $\operatorname{im}(p_n^*) \subseteq \ker(C^n(\iota))$. The composition $C^n(\iota)p_n^* = (p_n C_n(\iota))^*$ is the dual of a zero homomorphism and is zero, so $\operatorname{im}(p_n^*) \subseteq \ker(C^n(\iota))$.

3. We have inclusion $\ker(C^n(\iota)) \subseteq \operatorname{im}(p_n^*)$. Kernel $\ker(C^n(\iota))$ consists of group homomorphisms $\varphi \in C^n(X, A)$ with $C^n(\iota)(\varphi) = \varphi C_n(\iota) = 0$. By definition of cokernel of $C_n(\iota)$, there exists a map $\tilde{\varphi} \in C^n(X, U, A)$ with $p_n^*(\tilde{\varphi}) = \tilde{\varphi} p_n = \varphi$.

4. $C^n(\iota)$ is epi, because any group homomorphism $C_n(U) \to A$ can be extended with zero to $C_n(X) \to A$.

∎

After obtaining the cochain complex $C^\bullet(X, U, A)$, we can take its cohomology groups, which we denote by $H^n(X, U, A)$. This is a version of relative cohomology, known from algebraic topology, in the context of small categories.

Lemma 29. Let $U \xrightarrow{\iota} X$ be the inclusion of a subcategory inside $X \in \mathrm{Cat}$. For $n \geq 0$,

$$H^n(X, U, A) \xrightarrow{H^n(p_n^*)} H^n(X, A) \xrightarrow{H^n(C^n(\iota))} H^n(U, A)$$

is exact; in other words, $\mathrm{im}(H^n(p_n^*)) = \ker(H^n(C^n(\iota)))$.

Proof. For a short exact sequence of chain complexes $0 \to E \to E' \to E'' \to 0$, there is a standard construction of a long exact sequence. We apply this construction to

$$0 \to C^\bullet(X, U, A) \xrightarrow{p^*} C^\bullet(X, A) \xrightarrow{C^\bullet(\iota)} C^\bullet(U, A) \to 0.$$

We recognise certain terms in the resulting long exact sequence:

$$H^n(X, U, A) \xrightarrow{H^n(p_n^*)} H^n(X, A) \xrightarrow{H^n(C^n(\iota))} H^n(U, A).$$

∎

Lemma 30. Let $U \xrightarrow{\iota} X$ be the inclusion of a subcategory. Let $n \geq 0$ be an integer and $\omega \in \ker(H^n(\iota, A))$. There exists a representative ϵ of ω with $\epsilon \in \ker(C^n(\iota))$. In particular, if X is connected and U is contractible in X, then $\tilde{H}^n(X, A) \subseteq \ker(H^n(\iota, A))$. Therefore, for $\omega \in \tilde{H}^n(X, A)$ there exists $\epsilon \in \ker(C^n(\iota))$.

Proof. We chase around the following diagram

$$\begin{tikzcd}
\ker(b_n^{X/U}) \arrow[r] \arrow[d] & \ker(b_n^X) \arrow[r] \arrow[d] & \ker(b_n^U) \arrow[d] \\
H^n(X, U, A) \arrow[r] & H^n(X, A) \arrow[r] & H^n(U, A)
\end{tikzcd}$$

where the vertical arrows are projections. ∎

It is standard that the hom cofunctor $\hom_{\mathrm{Ab}}(\bullet, A)$ takes direct sums into products; in particular we have an isomorphism of abelian groups

$$\prod_{n\geq 0} C^n(X, A) \cong \hom_{\mathrm{Ab}}(\bigoplus_{n\geq 0} C_n(X), A).$$

There is an injective map

$$\bigoplus_{n\geq 0} C^n(X, A) \hookrightarrow \prod_{n\geq 0} C^n(X, A).$$

Putting this together, we have an injective map

$$\bigoplus_{n\geq 0} C^n(X, A) \hookrightarrow \hom_{\mathrm{Ab}}(\bigoplus_{n\geq 0} C_n(X), A).$$

We use this to identify elements of the left hand side (source) with certain group homomorphisms; in other words, to each element $\omega \in \bigoplus_{n\geq 0} C^n(X, A)$ there is a corresponding group homomorphism $\omega : \bigoplus_{n\geq 0} C_n(X) \to A$, which we will also denote by ω.

Suppose now that U is a subcategory of X. Recall that $C_n(X)$ by definition is the free abelian group on the set of functors of $[n]$ into X. Similarly, define $C_n(U)$ to be the free abelian group on the set of functors of $[n]$ into U. Since U is a subcategory of X, we may view each functor σ of $[n]$ into U as a functor of $[n]$ into X. In this way, we identify the set of functors of $[n]$ into U with functors of $[n]$ into X. We view the set of functors of $[n]$ into U as a subset of the set of functors of $[n]$ into X. Furthermore, we view $C_n(U)$ as a subgroup of $C_n(X)$. Finally, we may view the direct sum $\bigoplus_{n\geq 0} C_n(U)$ as a subgroup of $\bigoplus_{n\geq 0} C_n(X)$ (which it is, depending on the set theoretic construction we use to construct our direct sums).

Therefore it makes sense for $\omega \in \bigoplus_{n\geq 0} C^n(X,A)$ to write $\omega_{|\bigoplus_{n\geq 0} C_n(U)} = 0$, when U is a subcategory of X. $\bigoplus_{n\geq 0} C^n(X,A)$ can be made into a graded ring with the product $\star$. In fact, $\bigoplus_{n\geq 0} \ker(b_n)$ is a subring of $\bigoplus_{n\geq 0} C^n(X,A)$ and $H^*(X)$ is obtained as quotient of $\bigoplus_{n\geq 0} \ker(b_n)$ by a homogeneous ideal $\bigoplus_{n\geq 0} \operatorname{im}(b_{n-1})$.

Lemma 31. Let $\{U_j | 1 \leq j \leq m\}$ be a finite set of subcategories of X and let $\{x_j | 1 \leq j \leq m\} \subseteq \bigoplus_{n\geq 0} C^n(X,A)$, with the additional assumption that $x_{j|\bigoplus_{n\geq 0} C_n(U_j)} = 0$ for each $1 \leq j \leq m$. Then $(x_1 \cdot \dots \cdot x_m)_{|\sum_{j=1}^m \bigoplus_{n\geq 0} C_n(U_j)} = 0$.

Proof. $(x_1 \cdot \dots \cdot x_m)_{|\sum_{j=1}^m \bigoplus_{n\geq 0} C_n(U_j)} = 0$ is true if and only if for any integer $n \geq 0$, for any $1 \leq j \leq m$ and for any $[n] \xrightarrow{\sigma} U_j$, we have $(x_1 \cdot \dots \cdot x_m)(\sigma) = 0$. This is evident, after we write out $x_1 \dots x_m$. ∎

Lemma 32. Suppose $\{U_j | 1 \leq j \leq m\}$ is a finite geometric cover of $X \in \mathrm{Cat}$ and $\{x_j | 1 \leq j \leq m\} \subseteq \bigoplus_{n\geq 0} C^n(X)$ with the property that $x_{j|\bigoplus_{n\geq 0} C_n(U_j)} = 0$, $1 \leq j \leq m$. Then $x_1 \cdot \dots \cdot x_m = 0$.

Proof. By Lemma 31, $(x_1 \cdot \dots \cdot x_m)_{|\sum_{j=1}^m \bigoplus_{n\geq 0} C_n(U_j)} = 0$ and we claim that

$$\sum_{j=1}^m \bigoplus_{n\geq 0} C_n(U_j) = \bigoplus_{n\geq 0} C_n(X).$$

This is true, provided that for any $n \geq 0$ and any $\sigma : [n] \to X$, $\sigma \in \sum_{j=1}^m \bigoplus_{n\geq 0} C_n(U_j)$. This is guaranteed by the fact that $\{U_j\}_{j=1}^m$ is a geometric cover. ∎

We now prove our main result.

Theorem 33. Let $X \in \mathrm{Cat}$ be connected. If there exists a finite geometric cover of X with subcategories, contractible in X, then $\mathrm{cl}(X)$ is finite and $\mathrm{cl}(X) < \mathrm{cat}(X)$.

Proof. We go by following these steps:

1. Suppose $\{U_j | 1 \leq j \leq m\}$ is a finite geometric cover of X with U_j contractible inside X. We have to show that all products of m factors in $\tilde{H}^*(X,A)$ vanish.

2. $\tilde{H}^*(X,A)$ is a homogeneous ideal in $H^*(X,A)$ so products of m elements of $\tilde{H}^*(X,A)$ vanish if and only if all products of m homogeneous elements of $\tilde{H}^*(X,A)$ vanish. So now let $\omega_1, \dots, \omega_m \in \tilde{H}^*(X,A)$ be homogeneous elements.

3. By Lemma 30, for any $n \geq 0$, any homogeneous element $\omega \in \tilde{H}^n(X,A)$ and for any $j \in \{1, \dots, m\}$, there exists a representative $\epsilon \in \ker(b_n)$ with $\epsilon_{|\bigoplus_{k\geq 0} C_k(U_j)} = 0$. In particular, there exist ϵ_j representatives for ω_j with $\epsilon_{j|\bigoplus_{k\geq 0} C_k(U_j)} = 0$ for all $1 \leq j \leq m$.

4. Then $\omega_1 \cdot \dots \cdot \omega_m = [\epsilon_1] \cdot \dots \cdot [\epsilon_m] = [\epsilon_1 \dots \epsilon_m]$. By Lemma 31 and Lemma 32, we have $\epsilon_1 \dots \epsilon_m = 0$, which implies $\omega_1 \cdot \dots \cdot \omega_m = 0$. ∎

In Theorem 33, above, we showed that for any connected small category X, we have that the cup-length, defined through the cohomology of X, is a lower bound on its Lusternik–Schnirelmann category. It is worthwhile to note that all our arguments work only with the simplicial set $N(X)$, the nerve of the (small) category X in question. In future work, we suspect that results of this type may admit a unified view through the use of ∞-categories and the formulation of cohomology theories in that language.

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