BALLISTIC DEPOSITION AND KARDAR-PARISI-ZHANG UNIVERSALITY CLASS

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The paper focuses on modeling the growth of interface, best pictured as deposition of snow. In the first part a heuristic derivation of the Kardar-Parisi-Zhang (KPZ) equation is provided based on simple symmetry arguments. Next, the KPZ universality class is defined through an example of a simple discrete model, called Ballistic deposition. Numerical evidence is provided, which suggest that the KPZ equation is indeed in the KPZ universality class. Lastly, an experimental realisation is described, which shows characteristic KPZ statistics in boundary growth in liquid crystals.

BALISTIČNO ODLAGANJE IN KPZ UNIVERZALNOSTNI RAZRED

Članek se osredotoči na modeliranje rasti mejne plasti, kot na primer pri odlaganju snega. V prvem delu je preko preprostih simetrijskih argumentov predstavljena hevristična izpeljava Kardar-Parisi-Zhang (KPZ) enačbe. Nato je predstavljen preprost diskreten model, preko katerega je definiran KPZ univerzalnostni razred. Predstavljeni so numerični dokazi, ki nakazujejo, da KPZ enačba spada v KPZ univerzalnostni razred. Na koncu je razložen preprost eksperiment, ki pokaže značilno KPZ statistiko v rasti mejne plasti v tekočih kristalih.

1. Introduction

Imagine it's winter. You're inside, sitting comfortably next to the radiator, watching through the window as snow falls. Outside you see snowflakes as they slowly slide down on your window and aggregate on top of the ledge. You see a peculiar pattern and decide to take a close-up photo to investigate further. You see that at length scales comparable to the size of the snowflakes, the aggregate is very rough (Fig.1). Why? Snowflakes are deposited randomly. Once they arrive at the aggregate they stick. The randomness in the deposition process apparently leads to a rough surface.

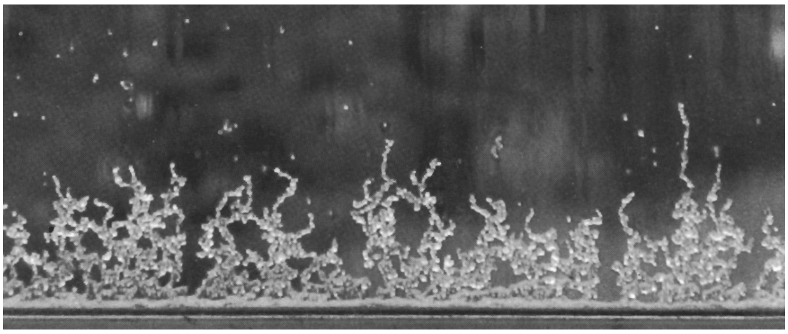

Figure 1. Snow aggregating on top of a window ledge. Figure taken from [1].

In this paper we aim to present the KPZ equation which models growth in such processes. It was named after the physicists Kardar, Parisi and Zhang [2], who, by studying it, unravelled an entirely new and rich field of study of universality classes. But what is a universality class? Roughly speaking, it is a collection of physical and mathematical models which are described by the same statistics and obey common scaling relations. For example, for a large number of systems we find that the roughness of the surface, $w(t)$, increases as a power of time, $w(t) \sim t^{\beta}$. It eventually saturates at a value that increases as a power of the system size: $w(L) \sim L^{\alpha}$. Studying such scaling relations allows us to define universality classes. But more on that later. Let us first try to "deriv" the KPZ equation.

2. The KPZ equation and stochastic growth

As it turns out, stochastic differential equations are a very successful tool for understanding the behaviour of various growth processes. One of the initial tasks of investigating a new phenomenon is to attempt to derive a continuum equation by considering simple symmetries. Consider a $(1+1)$ dimensional model. Suppose $h(x,t)$ describes the height profile of deposition at site $x \in \mathbb{R}$ and time $t \in [0,\infty)$. In general the growth can be described by the continuity equation

$$\partial_t h(x,t) = \Phi(x,t), \tag{1}$$

where $\Phi(x,t)$ is the number of particles per unit time arriving on the surface at position x and time t. The particle flux is not uniform, since the particles are deposited at random positions. This can be incorporated by decomposing Φ into two terms, so that equation (1) becomes

$$\partial_t h(x,t) = G(x,t,h,\partial_x h,(\partial_x)^2 h,\dots) + \eta(x,t).$$

The first term, $G(x,t,h,\partial_x h,(\partial_x)^2 h,\dots)$ is a function that depends on time, position, the interface height, and on its derivatives. The second term, $\eta(x,t)$, reflects the random fluctuations in the deposition process. It is an uncorrelated random number with zero average:

$$\langle \eta(x,t) \rangle = 0 \tag{2}$$

and the second moment is

$$\langle \eta(x,t)\eta(x',t') \rangle = 2D\delta(x-x')\delta(t-t'), \tag{3}$$

where D is some positive constant. Equation (3) implies that the noise has no correlation in space and time. The next step would be to consider the desired symmetries of the growth system:

- *Invariance under time translation.* The growth should not depend on where we define the origin of time, which means that, if $h(x,t)$ solves the equation, then so must $h(x,t+\delta t)$. Meaning that G does not explicitly depend on t.

- *Translation invariance along the growth direction.* The growth should be independent of where we define $h=0$, so if $h(x,t)$ solves the equations, then so must $h(x,t)+\delta h$. Meaning that G does not explicitly depend on $h(x,t)$ and so it must be constructed from higher derivatives $\partial_x^n h(x,t)\,, n \geq 1$.

- *Ivariance under spatial translation.* The equation should not depend on the actual value of x. So if $h(x,t)$ solves it, then so must $h(x+\delta x,t)$. Meaning that G does not explicitly depend on x. Note that all of the spatial derivatives $\partial_x^n h(x,t)$ respect this symmetry, and are therefore not removed at this step.

- *Rotational symmetry about the growth direction.* In one dimension, this results in spatial inversion symmetry. Thus if $h(x,t)$ solves the equation, then so must $h(-x,t)$. This removes all terms of odd degree of derivative, e.g., $(\partial_x^n h(x,t))^m$, where $m \times n$ is odd.

The possibly simplest such equation is the following linear partial differential equation, called the Edward-Wilkinson (EW) equation:

$$\partial_t h(x,t) = \nu \partial_x^2 h(x,t) + \eta(x,t), \tag{4}$$

where ν is a constant parameter and corresponds to surface tension, which tends to smooth irregularities of the interface.

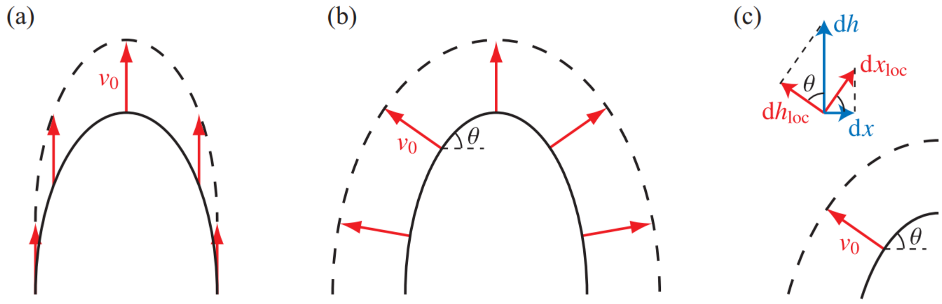

Figure 2. (a): interface growth according to the EW equation. (b): growth of the interface in the locally normal direction. (c): connection between local, $(h_{\rm loc}, x_{\rm loc})$, and global, (h, x), coordinates. Figure taken from [3].

However, as it turns out, this equation isn't sufficiently generic; growth processes in nature are in fact usually nonlinear. Equation (4) implicitly assumes the interface would always grow along the h-axis, Fig.2(a). This implies that the interface would attempt to advance upward with speed, say ν_0, even though it will also be deformed by noise and diffusion of the EW equation. In contrast, what happens more naturally is that each local piece of the interface advances in the direction normal to the interface, Fig.2(b). To take this effect into account, we assume that the interface is evolved locally by the EW equation, with height and space coordinates $(h_{\rm loc}, x_{\rm loc})$ taken locally along the normal and tangential axis, respectively. Then we transform $(h_{\rm loc}, x_{\rm loc})$ to global coordinates (h, x) and obtain a new equation. Fig.2(c) illustrates how we do this by connecting infinitesimal changes $({\rm d}h, {\rm d}x)$ and $({\rm d}h_{\rm loc}, {\rm d}x_{\rm loc})$ in the following way

$$ {\rm d}h = {\rm d}h_{\rm loc}/\cos\theta = {\rm d}h_{\rm loc}\sqrt{1+(\partial_x h)^2}, \quad {\rm d}x = {\rm d}x_{\rm loc}\cos\theta = {\rm d}x_{\rm loc}/\sqrt{1+(\partial_x h)^2}, $$

where θ is the local inclination angle, hence $\tan\theta = |\partial_x h|$. By inserting it to the evolution of $(h_{\rm loc}, x_{\rm loc})$, given by the EW equation (4) and keeping only the lowest order nonlinear terms of $\partial_x h$, which respect the above described symmetries, we finally arrive at the KPZ equation

$$ \partial_t h(x,t) = \nu \partial_x^2 h(x,t) + \frac{\lambda}{2}(\partial_x h(x,t))^2 + \eta(x,t). \qquad (5) $$

In next chapter we will see that the nonlinear term was indeed necessary to describe growth of interface. It is responsible for the lateral growth of interface seen in nature such as in, for example, snow (Fig.1).

3. Discrete models

The KPZ equation (5) is a stochastic partial differential equation, which means that its solution $h(x,t)$ is not a continuous function which would describe our growth model. Its actual solution is a continuously indexed family of random variables $\{h(x,t)\}_{x,t}$ and their respective family of probability distributions. The KPZ equation inspired an avalanche of studies of various mathematical models with similar statistical properties, thus coining the term KPZ universality class. Funnily enough, the KPZ equation itself had not been proven to actually be in this class until 20 years after its first discovery [4]. Furthermore, presenting any rigorous analytic treatment is beyond the scope of this paper. Thus for further discussion we limit ourselves to numerical integration. But before doing so, let us turn our attention to a much simpler, discrete model and use it to more precisely define the meaning of the KPZ universality class.

3.1 Ballistic deposition

Ballistic deposition (BD) is a $(1+1)$ dimensional lattice model, first proposed as a simple model for sedimentation. As it is shown in figure 3, we consider square particles of unit size, which are dropped onto a one-dimensional lattice of length $L \in \mathbb{N}$, with periodic boundary conditions. The particles are dropped onto random lattice sites at a constant rate, one at a time. Each particle falls vertically and sticks upon first contact with another particle. Mathematically speaking, we study the growth process of the height profile $h(j,n) \in \mathbb{N}_0$, given by the equation

$$h(j,n+1) = \begin{cases} \max\left[h(j-1,n),\, h(j,n)+1,\, h(j+1,n)\right], & \text{if } j=i \\ h(j,n), & \text{otherwise,} \end{cases} \tag{6}$$

where $i \in \mathbb{Z}_L$ is the randomly chosen site on the lattice onto which the particle falls at time $n \in \mathbb{N}_0$.

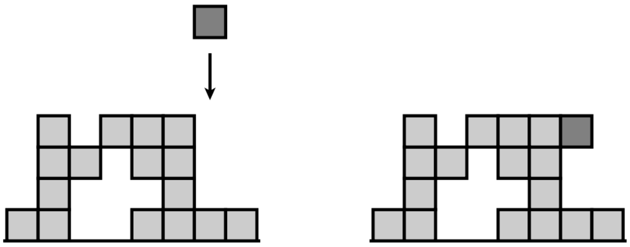

Figure 3. Left: Shows blocks falling onto randomly chosen lattice site. Right: They stick to the highest neighbouring block. Figure taken from [5].

You may be wondering what is the connection between this model and the KPZ equation? As it turns out KPZ equation can be derived as a continuous limit of BD [6]. To be more precise, some studies suggest [7], that the more accurate continuum limit of BD model is slightly different from the KPZ equation. Namely, equation (5) changes by the substitution of the square of the derivative $(\partial_x h)^2$ for the absolute value of the derivative $|\partial_x h|$. This however, does not change the behaviour of the solution, as it has been shown that in principle we could replace this term with any even power of the derivative and still get the same qualitative results [8]. This already hints to some universal behaviour of growth models. Furthermore, BD satisfies all the symmetry requirements we inferred for the KPZ equation, thus it is reasonable to assume to have similar behaviour.

One further notice: The growth rules (6) are most easily formulated with a time unit of 1 corresponding to one particle dropped. However, for simulations, it is more convenient to measure time in units of layers deposited. So in the proceeding figures, time is defined as $t := N/L$, where N is the total number of deposited particles. The reason for this is that $h(x,t)$ changes noticeably only when order of L particles is dropped; this is especially true for systems of large length L.

A picture of the evolution of the height profile $h(x,t)$ is shown in figure 4. We see that the deposited particles form a cluster with a very particular geometry, similar to the one observed in snow (Fig.1). The boundaries of different colours correspond to the height profile $h(x,t)$ at even and odd multiples of a chosen period. We see that the growth speed is approximately constant. To describe this more quantitatively, we introduce the *average height* of the surface

$$\bar{h}(t) := \frac{1}{L}\sum_{x=1}^{L} h(x,t).$$

As shown in figure 4, it grows linearly in time and is independent of system size. Together with constant deposition rate it follows that the density of the bulk is constant. Furthermore, we define the *interface width of a single realisation of the simulation*

$$w_i(L,t) := \sqrt{\frac{1}{L}\sum_{x=1}^{L}\left(h(x,t)-\bar{h}(t)\right)^2}, \qquad (7)$$

which quantifies the roughness of the interface at time t. However, it turns out that it fluctuates strongly. Therefore, the relevant quantity is actually the *interface width*, the average of (7) over M different realisations of the growth process

$$w(L,T) := \frac{1}{M}\sum_{i=1}^{M} w_i(L,t). \qquad (8)$$

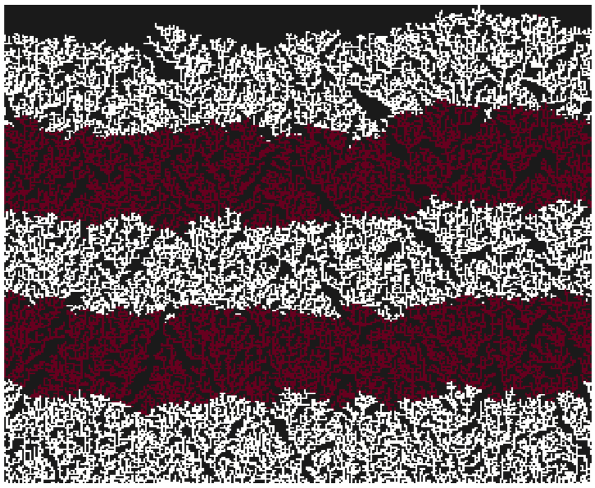

Figure 4. Simulation of BD model for lattice of length $L = 220$ run for 2×10^4 deposition steps. The red and white colours represent different multiples of a chosen period of deposition ($T = 4000$).

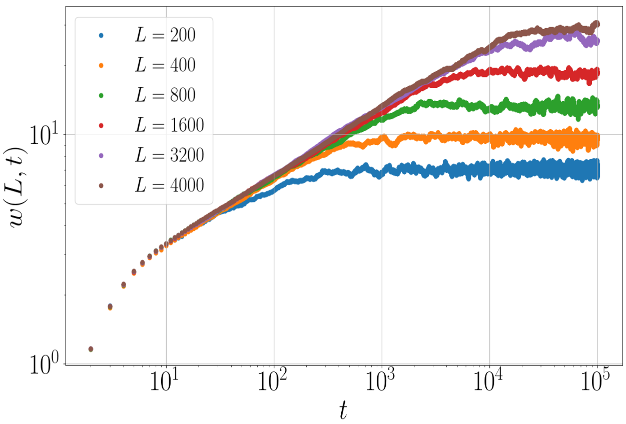

Figure 5. Interface width of BD for systems of different lengths L. Each curve was computed with an average over $M = 100$ realizations of simulations.

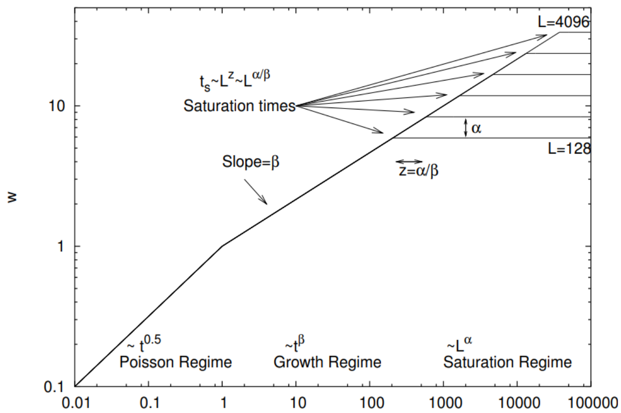

Figure 6. Figure shows three characteristic regimes of growth of interface width of BD for systems of different lengths, starting from $L = 128$, doubling up to $L = 4096$. Figure taken from [9].

Each of the curves $w(t)$ in figure 5 can be divided into three main regimes. There is an initial transition regime, $t \in (0, \sim 10)$, which corresponds to the first few layers of deposition, where there is a high probability that a new-coming particle will not stick to another block. This is essentially equivalent to BD deposition without sticking. It is followed by the growth regime, which has a power law growth

$$w(L,t) \sim t^{\beta}, \quad t \in (\sim 10, t_s). \qquad (9)$$

This lasts up to a certain time t_s, upon when the interface width saturates

$$w(L, t \geq t_s) = w_\infty(L) \qquad (10)$$

The quantity $w_\infty(L)$ is, appropriately, called saturation width. Furthermore, the saturation widths are approximately spaced equidistant in the log-log plot for each doubling of L, Fig.5. This suggest power law dependence of the saturation width on L

$$w_\infty(L) \sim L^{\alpha}. \qquad (11)$$

The time needed to reach the saturation regime also depends on L. Again figures 5 and 6 suggest the power law

$$t_s \sim L^z. \tag{12}$$

The exponents α, β and z are the ones which were alluded to in the introduction. α is called the roughness exponent and controls how the saturated interface scales with system size, β is called the growth exponent and controls how the interface roughness increases with time and z is called the dynamic exponent and tells us how the correlation of height spreads through out the interface. In fact, the exponent z is related to α and β. Approaching t_s from the right and from the left we expect to get the same value

$$t_s^\beta = \lim_{t\to t_s^-} w(L,t) = \lim_{t\to t_s^+} w(L,t) = L^\alpha, \tag{13}$$

thus, we get from Eq. (11) and (12)

$$z = \alpha/\beta. \tag{14}$$

It was first discovered by Family and Vicsek [10] that one can summarize equations (9), (10) and (11) by the finite-size scaling relation

$$w(L,t) \propto L^\alpha f(t/L^z), \quad f(x) \sim \begin{cases} x^\beta, & x \ll 1 \\ \text{const.}, & x \gg 1 \end{cases}. \tag{15}$$

This implies that plotting $w(L,t)/L^\alpha$ vs. $t/L^{\alpha/\beta}$ should collapse all $w(L,t)$ onto the same curve if the right exponents α, β are used. The exponents stated in literature for BD are $\alpha = 1/2$ and $\beta = 1/3$ [1, 11]. Figure 7 shows the attempted data collapse is successful for larger systems, while we still see deviation from the expected scaling in the transient region.

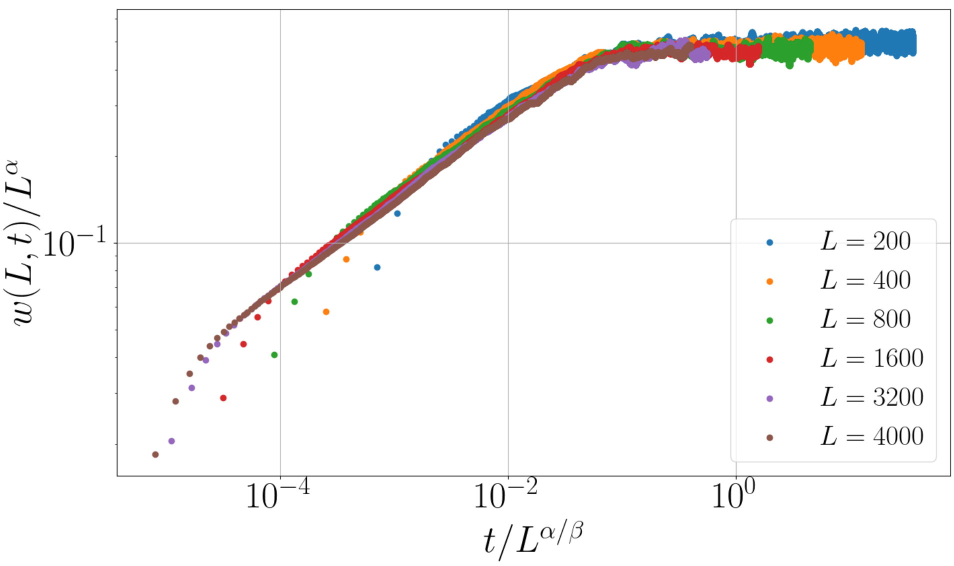

Figure 7. Data collapse of BD interface width from simulations according to the Family-Vicsek scaling relation (15) for exponents $\alpha = 1/2$ and $\beta = 1/3$. Data is in accordance with that of figure 5.

We may now finally define what is a universality class. It is a collection of physical and mathematical models, which share the same scaling exponents described above. In the case of the KPZ universality class, they are $\alpha = 1/2$, $\beta = 1/3$ and $z = 3/2$ [12]. Table 1 suggest that numerical values converge to the ones expected as we increase the system length L. From these, although imperfect, numerical results and successful data collapse (Fig.7), we may conclude that BD is indeed in the KPZ universality class.

Table 1. Growth and roughness exponents for different system size of BD model. All were calculated in accordance with data shown on figure 5. β was calculated by fitting the slope on to interface width; α was calculated by fitting saturation widths versus system lengths.

System length	α	β
$L = 1200$		0.268
$L = 1600$		0.283
$L = 2400$	0.472	0.292
$L = 3200$		0.299
$L = 4000$		0.300
Expected	1/2	1/3

3.2 The discretized KPZ equation

We may now put our attention towards the KPZ equation, showing that it is indeed in the KPZ universality class. We do this through numerical integration using the finite difference method, and by approximating the time and spatial derivatives in standard way we get

$$\begin{aligned} h(x,t+\Delta t) =& h(x,t) + \nu \frac{\Delta t}{\Delta x}\left[h(x+\Delta x,t) - 2h(x,t) + h(x-\Delta x,t)\right] \\ &+ \frac{\lambda}{8}\frac{\Delta t}{\Delta x^2}\left[h(x+\Delta x,t) - h(x-\Delta x,t)\right]^2 + \sigma\sqrt{12\Delta t}\zeta, \end{aligned}$$

where $\sigma := 2D/\Delta x$ and ζ is a random variable distributed uniformly on the interval $[-0.5, 0.5]$, which satisfies the equations (2) and (3). For practical purposes it is useful to use dimensionless variables $\bar{h} := h/h_0$, $\tau := t/t_0$ and $r := x/x_0$, where the characteristic units can be obtained from the coefficients of the growth equation $h_0 := \nu/\lambda$, $t_0 := \nu^2/\sigma^2\lambda^2$ and $x_0 := (\nu^3/\sigma^2\lambda^2)^{1/2}$. This is effectively setting $\nu = 1$, $\lambda = 1$ and $\sigma = 1$ in the above equation. Moreover, showing that the dimensionless KPZ equation satisfies the KPZ statistic shows that equation (5) belongs to the KPZ universality class no matter the underlining microscopic phenomena, i.e. irrespective of the coefficients ν and λ.

It is worth commenting that the discrete integration of the KPZ equation presents instabilities that make simulations diverge. The analysis performed by [13], shows that instabilities are caused by the nonlinear term $(\partial_x h)^2$, with or without the noise. To rectify this, we choose our time step such that $\Delta\tau \ll \Delta r$. However, this only makes the instabilities less probable. A possible solution is to introduce a cutoff ε, thus replacing the nonlinear term by

$$\min(\varepsilon, \left[h(r+\Delta r,\tau) - h(r-\Delta r,\tau)\right]^2).$$

In what follows, we take $\varepsilon = 200$, as it has been shown in [13] to be least restrictive, while still producing stable results. Furthermore, when we derived equation (5), we assumed spatial invariance, which means that we must take periodic boundary conditions. This restriction is convenient for numerical simulations, however it becomes unimportant in the thermodynamic limit, i.e. as we take L to infinity.

Figure 8 shows growth and attempted data collapse of interface width (8) for wedge initial condition $h_0 = |x|/L$. We see power law growth and saturation, which were characteristic to BD in previous section. Numerical results presented in (Tab.2) show fast convergence of exponents to expected values with the system length L, suggesting that KPZ equation is in the KPZ universality class.

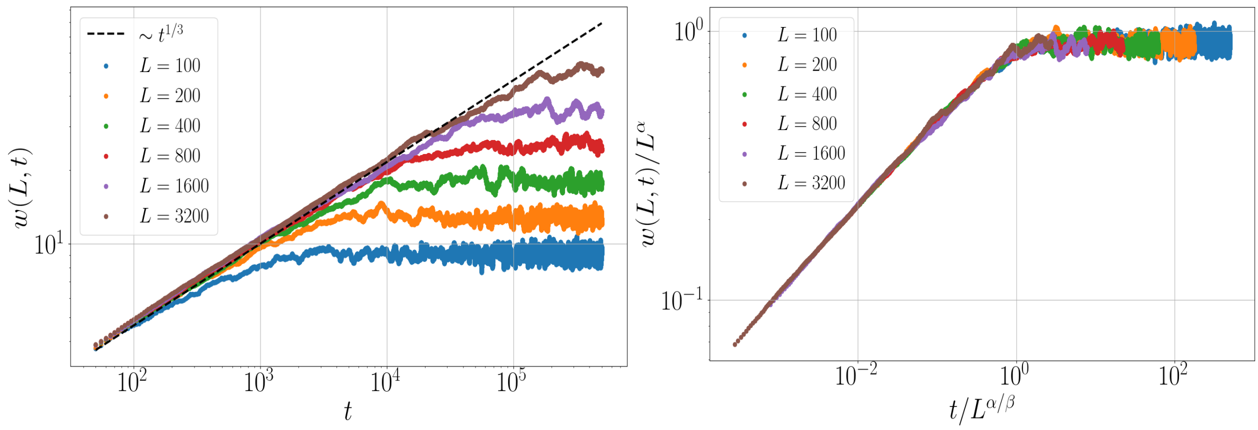

Figure 8. Left: Figure shows interface width of the discretized KPZ equation for systems of different lengths L. Each curve was computed with $M = 50$ realisations of simulations with the wedge initial condition $h_0 \sim |x|/L$ and discretisation $\Delta x = \sqrt{8\pi}$, $\Delta t = 1/2$. Right: Figure shows the associated data collapse using the Family-Vicsek scaling relation (15) for exponents $\alpha = 1/2$ and $\beta = 1/3$.

Table 2. Growth and roughness exponents for different system size of discretised KPZ equation. All were calculated in accordance with data shown in (Fig.8). β was calculated by fitting the slope on to the interface width; α was calculated by fitting saturation widths versus system lengths.

System length	α	β
$L = 800$		0.275
$L = 1600$	0.496	0.313
$L = 3200$		0.325
Expected	1/2	1/3

4. The probability distribution associated to KPZ universality

Historically, the wedge initial condition was the first for which it was rigorously proven that the KPZ equation belongs to the KPZ universality class [4]. It was shown that, for the wedge initial condition, the KPZ equation exhibits a transition of statistics. More precisely, for any position x and time t the solution to the wedge initial condition yields the probability distribution P, which satisfies the limiting behaviour

$$F_t(s) := P\left(h(x,t) - \frac{x^2}{2t} - \frac{t}{24} \geq -s\right)$$

$$F_t\left(2^{-1/3}t^{1/3}s\right) \xrightarrow{t\to\infty} F_{\mathrm{GUE}}(s), \quad F_t\left(2^{-1/2}\pi^{1/4}t^{1/4}\left(s - \log\sqrt{2\pi t}\right)\right) \xrightarrow{t\to 0} G(s),$$

where $G(s)$ is the cumulative Gaussian distribution and $F_{\mathrm{GUE}}(s)$ is the Gaussian unitary ensemble Tracy-Widom (GUE TW) distribution, which is characteristic of the KPZ universality class [12]. Thus, as it is shown in (Fig.9), the KPZ equation represents a mechanism for crossing over between two universality classes - the KPZ universality class and the EW (Edward Wilkinson) class at short-time, which is characterised by the Gaussian distribution and growth exponent $\beta = 1/4$ [12].

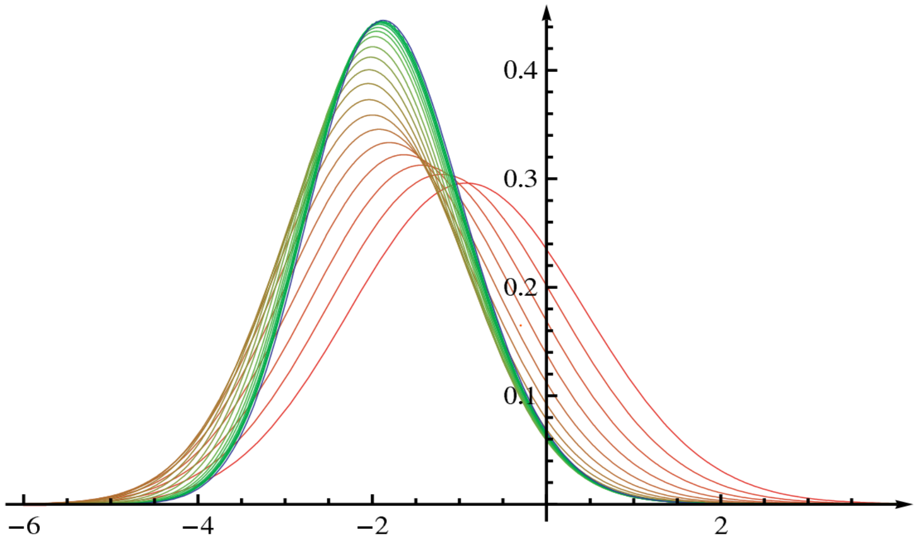

Figure 9. Figure shows the graph of the probability density function associated to $F_t(s)$, plotted for various values of t. As t increases we see the crossover from the graph of the Gaussian probability density distribution (red), to the GUE TW probability density distribution (green). Figure was taken from [12].

We now ask how the initial condition affects characteristic exponents and statistics for the long time fluctuations? As it turns out, the characteristic exponents remain unchanged, while the statistics, that is the height distributions, vary for different initial conditions [12]. For example, the limiting behaviour of probability distribution for the flat initial condition is described by the Gaussian orthogonal ensemble Tracy-Widom (GOE TW) distribution F_{GOE} [12, 14]. Moreover, more complicated initial data, corresponding to combinations of the wedge and flat initial condition, also give the same characteristic exponents, however its statistics are described neither by F_{GUE} or F_{GOE} [12].

5. An Experimental realisation

Let us conclude this paper by discussing an insightful experimental realisation of KPZ statistic. We consider convection of nematic liquid crystal, confined in a thin container, driven by an electric field. We focus on the interface of the boundary between two different states, one of which consists of large quantity of topological defects and can be created by nucleating a defect with an ultraviolet laser pulse. This state persists and grows constantly and radially, forming a cluster boarded by moving rough interface (Fig.10(a)), which effectively corresponds to interface growth for a single point initial condition. Alternatively, we can realise flat initial conditions as well (Fig.10(b)), simply by shooting a line shaped laser pulse through a cylindrical lens. In both cases, we see that the interface width exhibits power law growth with characteristic KPZ exponent $\beta = 1/3$ (Fig.12). Furthermore, the experiment permits us to study the long time probability distributions for each of the initial conditions. This can be seen by plotting a histogram of rescaled height $\chi = (h - \nu_\infty)/(\Gamma t)^{1/3}$, where ν_∞ and Γ are constant parameters. Just as we have explained in previous section, figure 11 suggest that interface exhibits different probability distribution for different initial data. In case of single point initial condition see the GUE TW distribution, while for the flat initial condition the GOE TW distribution.

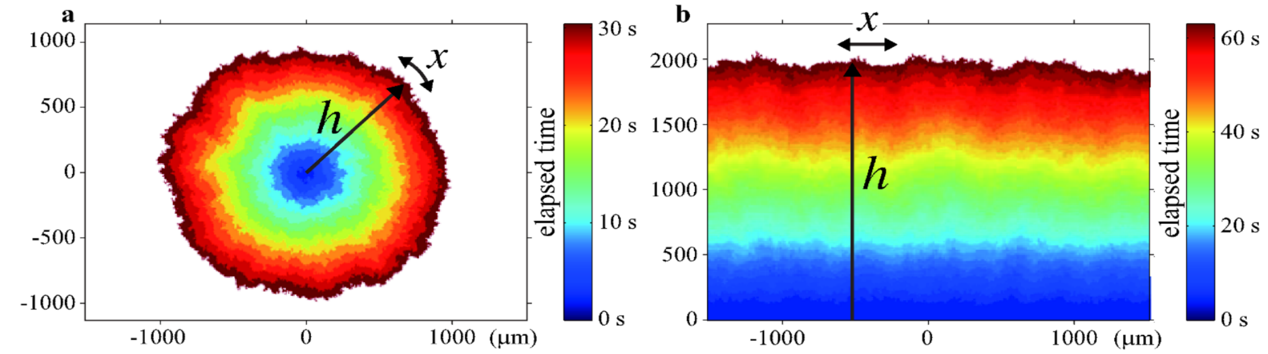

Figure 10. Growing cluster with circular (a) and flat (b) interface. Different colours represent snapshots of successive times. Indicated in the colour bar is the elapsed time after the laser emission. Local height $h(x,t)$ is defined in each case as a function of the lateral coordinate x along the mean profile of the interface. Figure taken from [15].

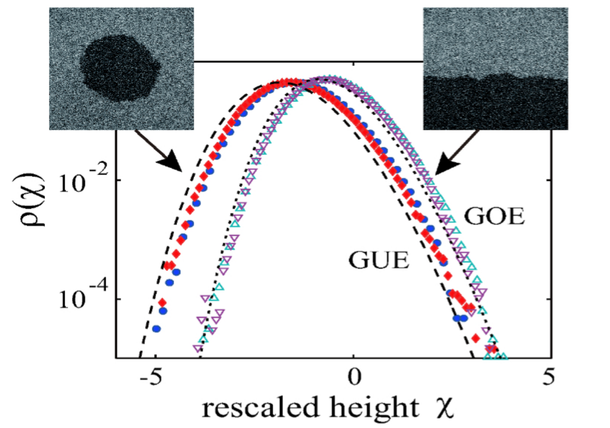

Figure 11. Histogram of the rescaled local height χ. The blue and red solid symbols show the histograms for the circular interfaces at $t = 10\,\mathrm{s}$ and $t = 30\,\mathrm{s}$; the light blue and purple open symbols are for the flat interfaces at $t = 20\,\mathrm{s}$ and $t = 60\,\mathrm{s}$, respectively. The dashed and dotted curves show the GUE and GOE TW distributions, respectively. Figure taken from [15].

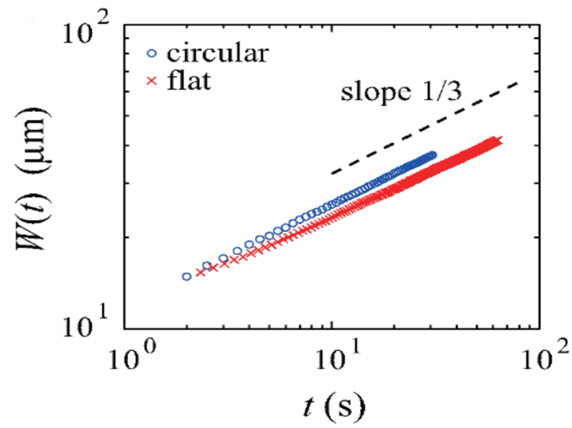

Figure 12. Growth of interface width for circular and flat interface. The dashed line is a guide for the eye showing the characteristic exponent $\beta = 1/3$. Figure taken from [15].

6. Conclusion

Motivated by our initial example of snow, we wrote down possibly the simplest equation respecting the required symmetries, Edward-Wilkinson equation. However, as it turned out it was not sufficiently general. To remedy this, we added an additional nonlinear term, which represented lateral growth, thus arriving at the famous KPZ equation. We then took a look at a related, although much simpler model, called Ballistic deposition, through which we defined exponents characteristic to the KPZ universality class. Finally we returned our attention to the KPZ equation providing numerical evidence that it also belongs to the KPZ universality class. However, as we remarked, this is not such a trivial fact, as it was only just proven in 2010 [4]. Lastly, we described an insightful experimental realisation of KPZ statistic in growth of boundary layer between two different states in nematic liquid crystal. We must remark that this paper has not even scratched the surface of the vast amount of studies performed on KPZ universality. For instance, KPZ statistics have recently been observed in seemingly unrelated spin models, such as the Heisenberg spin chain and its classical variants [16, 17]. These are of particular interest, since, as opposed to models of the interface growth, they describe physics in equilibrium. Although, as of right now this still remains conjectural and has yet to be rigorously proven.

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