ON HYDRODYNAMICS AS AN EFFECTIVE FIELD THEORY

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Hydrodynamics can be thought of as an effective theory which describes the long-distance behaviour of a system around thermal equilibrium. At an ever-increasing difficulty, it can be formulated as a derivative expansion in terms of hydrodynamic variables which are to be read off the partition function of the system in thermal equilibrium. The explanation of all this terminology is followed by a discussion on the truncation of hydrodynamics in the order of derivatives. Truncation at 0th order for normal relativistic fluids is discussed in more detail, which gives rise to 0th order hydrodynamics (ideal hydrodynamics) for a perfect fluid. It is noted that in the nonrelativistic limit, those equations become the usual equations of fluid dynamics. This is what is referred to as the modern formulation of hydrodynamics. Then, changing perspective, comments follow on high-energy (short-distance) and low-energy (long-distance) degrees of freedom, in the context of effective field theories. This is to inspire considering hydrodynamics itself as an effective field theory, when having its intrinsic nature in mind. Having motivated the discussion, the focus falls on formulating ideal hydrodynamics from an action principle. To that end, symmetries are pointed out, the thermodynamic quantities are identified, and finally it is shown how a specified equation of state restricts the form of the generic action. For obvious reasons, this is what is referred to as the more modern formulation of hydrodynamics.

O HIDRODINAMIKI KOT EFEKTIVNI TEORIJI POLJA

O hidrodinamiki se da razmišljati kot o efektivni teoriji, ki opiše obnašanje sistema okoli termičnega ravnovesja na dolgih razdaljah. Enega od načinov formulacije predstavlja razvoj po odvodih hidrodinamskih spremenljivk, katere je moč izračunati iz particijske funkcije sistema v termičnem ravnovesju. Po uvedbi terminologije sledi diskusija o redu hidrodinamike v relaciji na red najvišjih odvodov, ki so vključeni v razvoj. Podrobno je opisana hidrodinamika ničtega reda (idealna hidrodinamika) za normalne relativistične tekočine, kar poda opis idealnih tekočin. Po tem vzorcu je opredeljena moderna formulacija hidrodinamike. Sledijo komentarji o visokoenergijskih (kratke razdalje) in nizkoenergijskih (velike razdalje) prostostnih stopnjah v kontekstu efektivnih teorij polja. S tem je motiviran opis hidrodinamike kot efektivne teorije polja ob upoštevanju njene intrinzične narave. Sledi formulacije hidrodinamike iz principa minimizacije akcije. Ob upoštevanju relevantnih simetrij in identifikaciji termodinamskih količin je pokazano, kako enačba stanja definira obliko najbolj splošne akcije. Iz očitnih razlogov se temu formalizmu reče bolj moderna formulacija hidrodinamike.

1. Introduction

Liquids and gases are ubiquitous, but the theory of hydrodynamics (fluid dynamics) is even more so. This statement is key to inspiring our whole discussion. Presumably, there is much more to hydrodynamics than what catches the eye. When asked to imagine phenomena which would be governed by fluid dynamics, colloquially speaking, what may come to mind is the flow of water, the clouds and the wind in the sky, or perhaps even hurricanes. Indeed, simply upon agreeing to include liquids and gases under the same name of fluids, we agree that hydrodynamics consists of studying a remarkably rich set of phenomena. But it does not stop there. There are things that have not caught the eye for centuries without further enquiry, such as, the interior of neutron stars, the quark-gluon plasmas, or the cosmological universe with its galaxies being described as a fluid. Yet, those are still thought to be described by hydrodynamics. Or better say, we want to modify and expand hydrodynamics in such a way as to make room for all the new phenomena—not only for the new ones now in sight, but for the new ones yet to come. Such enquiries consisting of mathematical investigation and exotic applications have given rise to an enormous field of exciting research under the name of hydrodynamics [1, 2, 3]. To that end, there have been attempts to formulate hydrodynamics at an ever-increasing systematic procedure. This is what we will try to glimpse over in this work.

We begin in Section 2, where we introduce the modern definition of (relativistic) hydrodynamics as an effective theory. After explaining the terminology, we see that we have to identify the conserved quantities (Section 2.1) and the hydrodynamic variables (Section 2.2), in order to write down the constitutive relations (Section 2.3), which allows us to organize hydrodynamics as a derivative expansion. We note that the conserved quantities are the ones which describe the fluid at hand. In the same breath, we pick the ones we will use for normal relativistic fluids, and then briefly note the ones for superfluids and magnetohydrodynamics. The structural similarities between them is what makes this an elegant procedure. We then restrict the discussion to normal relativistic fluids, for which we pick the hydrodynamic variables and write the constitutive relations. That is where we note that the Euler equations and the Navier-Stokes equations follow as the nonrelativistic limit of 0th order and 1st order hydrodynamics, respectively. Then, in Section 3, we change perspective. We provide brief remarks about the short-distance and long-distance degrees of freedom in the context of effective field theories, which should inspire considering hydrodynamics as an effective field theory, due to its intrinsic nature of being an emergent theory with an underlying microscopic quantum field theory. Having motivated this end result, we restrict our attention to formulating ideal hydrodynamics from an action principle. In the spirit of effective field theories, we do that by first identifying the symmetries, and then writing the most general action consistent with those symmetries (Section 3.1). There, having the action enables us to derive the energy-momentum tensor. In Section 3.2, we identify all the thermodynamic quantities upon matching our derived energy-momentum tensor with the energy-momentum tensor for the perfect fluid. All the thermodynamic quantities end up being expressed in terms of the degrees of freedom of the action. Finally, in Section 3.3, we see how having an equation of state restricts the form of our generic action, which we note that in turn fixes all the thermodynamic quantities. We conclude in Section 4, with a few closing remarks.

2. Modern formulation of (relativistic) hydrodynamics

Hydrodynamics can be thought of as an effective theory which describes the long-distance behaviour of a system around thermal equilibrium [1, Section 1.1], [2, Chapter 2], [3, Appendix A].[1] Let us explain what we mean by this terminology. By an effective theory we mean that by the time hydrodynamics becomes applicable, the microscopic details have been averaged out; this coincides with long-distance behaviour, which means behaviour at low energies. And what we mean by microscopic details is the underlying microscopic theory which underlies the emergent behaviour; that is presumably a quantum field theory, as is the standard model of particle physics. The relevant variables at low energies are also referred to as infrared (IR) variables, in the usual analogy with light at low energies [3, Section IB]. Of course, the low-energy (emergent behaviour) scale can only be made more precise once we have the theory at hand. Finally, considering a system around thermal equilibrium means considering the standard description of equilibrium statistical physics for a quantum or a classical system, and then at least linear perturbations around equilibrium (linear response) [4], [1, Section 2]. In fact, with hindsight, this explains why hydrodynamics was robust even before quantum mechanics.

More precisely, the formulation of hydrodynamics consists of writing constitutive relations in a systematic derivative expansion. The constitutive relations denote the conserved quantities expressed in terms of hydrodynamic variables [1]. Then the claim is that the conservation laws are the

[1]The main line of discussion for this section will be based on the notes by Kovtun [1] and the references therein. This was one of the first more modern pedagogical reviews on hydrodynamics as advertised in Section 2. of this work. The book [2] is even more recent and a more comprehensive review of (relativistic) hydrodynamics and, in addition, its current phenomenological applications. On the other hand, from the point of view of [3] this formulation of hydrodynamics is outdated, in a precise sense, which we will discuss in Section 3.

equations of motion for hydrodynamics. This means we need to identify the conserved quantities (Section 2.1) and the hydrodynamic variables (Section 2.2), before we are finally able write the constitutive relations (Section 2.3).

2.1 The conserved quantities

The conserved quantities motivationally stem from the symmetries of the fundamental microscopic theory. For instance, according to Noether's theorem in classical field theory, a continuous symmetry gives rise to a conserved current [5, Section 3.3], [6, Section 1.2].[2] More precisely, the symmetries can be transformations of spacetime coordinates or of internal variables of the action, and invariance of action under those transformations gives rise to the corresponding conserved currents [9, Section 2.6.2].

Concretely, say, invariance of the action $S[\phi(x)]$ under constant (spacetime) translational symmetry, $x^\mu \to x^\mu + \delta a^\mu$, gives rise to the conservation of the energy-momentum tensor $T^{\mu\nu}$,

$$\partial_\mu T^{\mu\nu} = 0, \tag{1}$$

and invariance of the action $S[\phi(x)]$ under (internal) $U(1)$ symmetry, $\phi \to e^{i\alpha}\phi$, gives rise to the conservation of a vector current J^μ,

$$\partial_\mu J^\mu = 0. \tag{2}$$

We require full relativistic covariance, so besides the aforementioned translational symmetry, we also need to consider the rotational and boost symmetry, i.e. Poincaré invariance. Both together, they correspond to the conservation of $M^{\mu\nu\rho} = x^\mu T^{\nu\rho} - x^\nu T^{\mu\rho}$ [1],

$$\partial_\rho M^{\mu\nu\rho} = 0. \tag{3}$$

At first, it is not obvious to see what should be the conserved quantities which would give rise to hydrodynamics. From early on, naively, we can assert that we need to have (a) conservation of energy, (b) conservation of momentum and (c) conservation of 'matter' [10].[3] We know that both (a) and (b) are encoded in the 4-momentum $P^\mu = (E, \mathbf{p})$, which is itself encoded in the energy-momentum tensor $T_{\mu\nu}$, in covariant notation [11]. Namely, $P^\mu = \int \mathrm{d}^3x\, T^{0\mu}$. On the other hand, of course, the word 'matter' in (c) is ambiguous. What we shall mean by it, is the conservation of charge N, in a precise sense: for a $U(1)$ internal symmetry, that is indeed given by the aforementioned vector current J^μ. More precisely, the charge is defined as $N = \int \mathrm{d}^3x\, J^0$. This discussion makes it clear that we have to use Eqs. (1) and (2). But, again for the same reasons, we cannot leave out Eq. (3), if we require full relativistic covariance. However, we shall see that upon modification of our definition of the energy-momentum tensor, Eq. (3) provides no new information, and can be omitted. The energy-momentum tensor $T^{\mu\nu}$, obtained by this so-called Noether's procedure, in general is not symmetric, $T^{\mu\nu} \neq T^{\nu\mu}$. In this case, the three conservation equations above are all independent. However, in some cases, albeit in an ad-hoc way [12], the energy-momentum tensor can be made symmetric by adding to it a divergenceless term, which does not ruin the conservation

[2] In fact, originally, Noether stated two theorems (the translation of the original 1918 paper can be found in [7, Part I]). The first one, which is the widely spread one referred to as Noether's theorem, concerns itself with global symmetries; indeed, this it is usually found in the literature: a *global* continuous symmetry gives rise to a conserved current [5, 6]. On the other hand, the second one concerns itself with local (gauge) symmetries [8]. For a comprehensive account see [7].

[3] These are reasonable for the so-called normal relativistic fluids: fluids which upon taking the non-relativistic limit, lead to the Euler and Navier-Stokes equations (see Section 2.3). Other options to have in mind are superfluids or magnetohydrodynamics, which we discuss a bit below.

law. The more refined argument, constitutes another definition for the energy-momentum tensor: by variation of the action with respect to the inverse metric $g^{\mu\nu}$, namely [13, p. 146],[4]

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}. \tag{4}$$

This definition ensures that the energy-momentum is symmetric and it obeys the same conservation law (in flat space).[5] But now Eq. (3) is just a consequence in light of Eq. (1). Indeed, assuming Eq. (1) and the symmetricity of $T^{\mu\nu}$,

$$\begin{aligned}\partial_\rho M^{\mu\nu\rho} &= \partial_\rho x^\mu T^{\nu\rho} - \partial_\rho x^\nu T^{\mu\rho} = \delta^\mu_\rho T^{\nu\rho} - \delta^\nu_\rho T^{\mu\rho} \\ &= T^{\nu\mu} - T^{\mu\nu} = T^{\nu\mu} - T^{\nu\mu} = 0,\end{aligned}$$

is identically zero. In other words, Eq. (1) holds complete information about relativistic covariance, since $T^{\mu\nu}$ and $M^{\mu\nu\rho}$ are not independent. We finally have our equations of motion for normal relativistic hydrodynamics, namely, Eqs. (1) and (2).

To talk about relativistic superfluids, in a recent formulation, in addition to Eqs. (1) and (2), up to caveats which we will not discuss [14], one needs one more additional constraint. Namely, the additional equation [14, Section 2.2],

$$\partial_\mu \xi_\nu - \partial_\nu \xi_\mu = 0, \tag{5}$$

where ξ^μ denotes the local value of the gradient of the superfluid phase [14], which is related to the chemical potential μ via the 4-velocity u^μ, by the equation $\mu = \xi_\mu u^\mu$.[6] For the superfluid then, Eqs. (1), (2) and (5), would be the equations of motion holding information about the conserved quantities of the system.

In a recent formulation for relativistic magnetohydrodynamics [15], the stated equations of motion are [15, Section 3.1],

$$\nabla_\mu T^{\mu\nu} = F^{\nu\rho} J_\rho, \tag{6a}$$

$$\nabla_\mu J^\mu = 0, \tag{6b}$$

where ∇_μ denotes the geometric covariant derivative [13, Section 3.2]. Now the energy-momentum tensor $T^{\mu\nu}$ is defined by varying the generating functional $W[g_{\mu\nu}, A_\mu]$ [15] with respect to the metric $g_{\mu\nu}$, and the vector current J^μ is defined by varying $W[g_{\mu\nu}, A_\mu]$ with respect to the gauge field A_μ, whereby they automatically satisfy Eqs. (6) [15]. For completeness, let us simply note that $F^{\nu\rho} = F^{\nu\rho}[A^\nu]$. In yet another formulation of relativistic magnetohydrodynamics presumably relevant to plasma physics [16], where instead of the vector current one considers a tensor current, the equations of motion are [16, Section II],

$$\nabla_\mu T^{\mu\nu} = H^\nu{}_{\rho\sigma} J^{\rho\sigma}, \tag{7a}$$

$$\nabla_\mu J^{\mu\nu} = 0. \tag{7b}$$

The energy-momentum tensor $T^{\mu\nu}$ is defined by varying the total action $S[g_{\mu\nu}, b_{\mu\nu}]$ [16] with respect to the metric $g_{\mu\nu}$ and the tensor current $J^{\mu\nu}$ by varying $S[g_{\mu\nu}, b_{\mu\nu}]$ with respect to the gauge field

[4] We are using the mostly plus metric signature and g denotes the determinant of the metric, $g = \det g_{\mu\nu}$. Furthermore, we are working in absence of gravity, hence we write the S_m from [13] which denotes the matter part, simply S.

[5] We can also define the current J^μ in a similar way. Namely, $J^\mu = \delta\mathcal{L}_m/\delta A_\mu$, where $\mathcal{L}_m$ is the matter part of the Lagrangian, and A^μ is the gauge field [12, Eq. (3) and Eq. (202)]. Nevertheless, as opposed to the redefinition of the energy-momentum tensor, this will not provide us with anything new.

[6] Both the chemical potential μ and the 4-velocity u^μ are are to be discussed in Section 2.2.

$b_{\mu\nu}$, whereby they automatically satisfy Eqs. (7) [16]. For completeness, let us simply note that $H^{\nu}{}_{\rho\sigma} = H^{\nu}{}_{\rho\sigma}[b_{\rho\sigma}]$.

In all cases—normal fluids, superfluids, or magnetohydrodynamics—the conservation equations denoting the conserved quantities are the equations of motion for hydrodynamics. This is remarkably elegant. Next, it remains to express them in terms of the relevant hydrodynamic variables which we pick in Section 2.2. In what follows, we restrict our attention to normal relativistic fluids.

2.2 The hydrodynamic variables

In statistical physics language, consider a system that can exchange energy and particles with a reservoir. In other words, this system is represented by the (grand) canonical ensemble [4, Section 3.6]. Then, in conventional notation, the density operator ρ is given by,

$$\rho = \frac{1}{Z} e^{-\beta H + \mu N}, \qquad Z = \operatorname{tr} e^{-\beta H + \mu N}, \tag{8}$$

where Z denotes the (grand) partition function and what sits in the exponential is the Hamiltonian operator H next to the Lagrange multiplier $\beta = 1/T$ which denotes the inverse temperature, and the number operator N next to the Lagrange multiplier μ which denotes the chemical potential.

The claim is that the hydrodynamic variables are precisely the Lagrange multipliers that sit in the exponential. In addition, they are promoted into slowly varying functions dependent on spacetime, $\beta \to \beta(x)$ and $\mu \to \mu(x)$,[7] so that we consider local thermal equilibrium. This agrees with the fact that hydrodynamics is concerned with slight deviations from thermal equilibrium. Indeed, the fact that they are slowly varying, ensures that we can make the derivative expansion (see Section 2.3). As is evident from this discussion, this formulation of hydrodynamics must be consistent with statistical physics (thermodynamics). This will become even more evident after we discuss the perfect fluid, where we impose the laws of thermodynamics.

But note that the Lagrange multipliers in Eq. (8) are just scalar quantities. Recall that what we want to do is express the covariant quantities $T^{\mu\nu}$ and J^{μ} in terms of the hydrodynamic variables. As is the case now, the most general decomposition we would be able to write down would immediately involve derivatives, so there would be no 0th order hydrodynamics. For example, when expressing the energy-momentum tensor in terms of those hydrodynamic variables we would have to write down terms such as $T^{\mu\nu} \sim \partial^{\mu}\partial^{\nu}\beta$ or $T^{\mu\nu} \sim \partial^{\mu}\beta\,\partial^{\nu}\beta$ or $T^{\mu\nu} \sim \partial^{\mu}\mu\,\partial^{\nu}\mu$ and so on, all involving derivatives.[8] We must do better.

Consider now what we shall call the covariant density operator [17, Section 4],

$$\rho = \frac{1}{Z} e^{\beta_{\mu} P^{\mu} + \gamma N}, \qquad Z = \operatorname{tr} e^{\beta_{\mu} P^{\mu} + \gamma N}, \tag{9}$$

where P^{μ} is the 4-momentum operator, and N is the operator of the conserved charge. The Lagrange multipliers are now β_{μ} which we require to be a timelike vector[9] and γ which is a scalar. This would enable us to write the constitutive relations at 0th order and beyond. Indeed, since we now have a 4-vector, namely β^{μ}, when expressing the energy-momentum tensor in terms of these hydrodynamic variables we would be able to write down terms such as $T^{\mu\nu} \sim \beta^{\mu}\beta^{\nu}$ which does not involve derivatives. Nevertheless, before we continue with the decomposition we consider a further reparametrization of the covariant density operator [1] to make contact with the usual density matrix. In other words, we want to see how the covariant density matrix reduces to the usual density matrix. We write $\beta_{\mu} = \beta u_{\mu}$ and $\gamma = \beta\mu$, where β and μ are the aforementioned quantities,

[7] Throughout, we will use the notation, $f(x) \equiv f(x^{\mu}) \equiv f(t, \mathbf{x}) \equiv f(t, x^{i})$.

[8] See also Section 2.3.

[9] We want to consider timelike constituents of the fluid. In our choice of metric signature, we are saying that $\beta_{\mu}\beta^{\mu} < 0$.

but now $u_\mu = (u_0, \mathbf{u})$ represents the velocity of the fluid, and now we also state explicitly that $P^\mu = (H, \mathbf{p})$, where H is the Hamiltonian (previously denoted by E). The second term in the exponential of the covariant density matrix already agrees with the second term in the exponential of the usual density matrix, up to a redefinition. Let us talk about the first term. So far we have $\beta u_\mu P^\mu = -\beta u_0 H + \mathbf{u}\cdot\mathbf{P}$, which does not look promising. We impose a further condition. Since u^μ is already timelike, we normalize it, $u^\mu u_\mu = -1$. Such a 4-velocity would be $u^\mu = (1-\mathbf{v}^2)^{-1/2}(1, \mathbf{v})$, where $\mathbf{v}$ is the spatial velocity. This again does not do it. But now all is clear, if only $\mathbf{v} = 0$, we would get back the the usual density operator. This already has an interpretation: it means we are imposing a rest frame. Then, specialized to the rest frame [17], up to the appropriate definitions, the covariant density operator is nothing but the usual density operator. With that in mind, we keep our covariant density matrix. Let us collect our new definitions. Eq. (9) becomes,

$$\rho = \frac{1}{Z} e^{\beta u_\mu P^\mu + \beta\mu N}, \qquad Z = \operatorname{tr} e^{\beta u_\mu P^\mu + \beta\mu N}. \tag{10}$$

Now our Lagrange multipliers are βu_μ and $\beta\mu$. But we can always choose to work with the three quantities independently, and we remind ourselves that $\beta = 1/T$. We finally have our hydrodynamic variables which we promote to slowly varying functions of spacetime, namely, $T(x)$, $u_\mu(x)$, and $\mu(x)$. We will make use of them in Section 2.3.

To encapsulate, we are using our parameterized covariant density matrix given by Eq. (10) for further enquiry. First, we noted that we have a precise statement on how to know what are the relevant hydrodynamic variables of the theory: they are the Lagrange multipliers sitting in our density matrix, $T(x)$, $u_\mu(x)$, and $\mu(x)$. Second, as we mentioned explicitly in Section 2.1, the quantities P^μ and N which are now sitting in our density matrix are both incorporated in $T^{\mu\nu}$ and J^μ, respectively. Finally, to repeat the argument from Section 2.1, the reason why we have to write conversation laws for $T^{\mu\nu}$ and J^μ instead of P^μ and N is to have a full relativistic treatment.

2.3 Constitutive relations and truncation

Before we express the conserved quantities in terms of all the hydrodynamic variables, let us first note the general decomposition of their tensorial structure, given only a timelike vector u^μ. They can be decomposed into components which are transverse and longitudinal with respect to u^μ, using the projector [1],

$$\Delta^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu. \tag{11}$$

Then the most general decompositions take the following form [1],[10]

$$T^{\mu\nu} = A\, u^\mu u^\nu + B\, \Delta^{\mu\nu} + (q^\mu u^\nu + q^\nu u^\mu) + t^{\mu\nu}, \tag{12a}$$

$$J^\mu = C\, u^\mu + j^\mu. \tag{12b}$$

The coefficients A, B, and C are scalars. The vectors q^μ and j^μ are transverse with respect to u^μ. That is,

$$u_\mu q^\mu = 0, \qquad u_\mu j^\mu = 0.$$

And finally, the tensor $t^{\mu\nu}$ is symmetric, transverse, and traceless. That is,

$$t^{\mu\nu} = t^{\nu\mu}, \qquad u_\mu t^{\mu\nu} = 0, \qquad g_{\mu\nu} t^{\mu\nu} = 0.$$

Now all the quantities in the decomposition (except for u^μ itself, obviously), are to be considered as functions of our hydrodynamic variables, $\mu(x)$, $T(x)$, and $u^\mu(x)$, including their derivatives. One

[10] The detailed explanation for the decomposition can be found in [10, Eqs. (7)-(12) and Eqs. (19)-(22)]. Note that our projector $\Delta^{\mu\nu}$, is denoted there by $s^{\alpha\beta}$.

may already be tempted to start writing all the combinations. Nevertheless, writing all the possible combinations in one breath is not possible, as the number of combinations is infinite. Instead, we organize them in order of derivatives—hence the term derivative expansion. Truncating at 0th order gives 0th order hydrodynamics or ideal hydrodynamics [1]. Due to the fact that this effectively means all the derivatives are zero, we are considering fluid dynamics at equilibrium, which in fact describes a perfect fluid. Truncating at 1st order, gives 1st order hydrodynamics, or dissipative hydrodynamics [1].[11] This means we are considering linear perturbations around equilibrium.[12] At such truncations, our conservation laws, Eqs. (1) and (2), in the nonrelativistic limit, become the Euler equations and the Navier-Stokes equations, respectively.[13] It is due to this nonrelativistic limit, how the name 'normal' in 'normal relativistic fluids' can be understood. Truncating at 2nd order gives 2nd order hydrodynamics, which has no such robust nonrelativistic analogue. At each order, the procedure becomes progressively more difficult. In fact, the equations of 3rd order hydrodynamics have been written down only recently by [20], whereby a procedural algorithm has been provided. For completeness, we note that this so-called classical hydrodynamics neglects thermal fluctuations [1, Section 3.2]. The theory of hydrodynamics which includes such fluctuations is dubbed stochastic hydrodynamics [21], or fluctuating hydrodynamics [3].

In what follows, we only elaborate in more detail about the truncation at 0th order. In other words, we only keep terms without derivatives, and omit them otherwise.

Consider the energy-momentum tensor, Eq. (12a). The first term has no derivatives, and nor does the second one. We keep these two. But note that we also have to impose $A = A(\mu, T)$ and $B = B(\mu, T)$, since keeping u^μ would amount to keeping the derivative term $\partial_\mu u^\mu$. The third term has $q^\mu u^\nu$. The only possible no-derivative expression from this in terms of hydrodynamic variables could be $u^\mu u^\nu$, since considering the other hydrodynamic variables would have to involve derivatives, $\partial^\mu \mu$, $\partial^\mu T$. But such a term can be absorbed into the first term, since A is a generic function. Precisely the same argument holds for the fourth term, and indeed for the fifth (last) term. We omit them all. We are left with,

$$T^{\mu\nu} = A\, u^\mu u^\nu + B\, \Delta^{\mu\nu}. \tag{13a}$$

Now consider the vector current, Eq. (12b). Again, the first term has no derivative, so we keep it, imposing $C = C(\mu, T)$. To repeat the argument, the second term j^μ, can only be u^μ, since the other two hydrodynamic variables would have to involve derivatives, $\partial^\mu \mu$, $\partial^\mu T$. But that is eaten by the first term, so we simply omit it. We are left with,

$$J^\mu = C u^\mu. \tag{13b}$$

Here comes the key point. We want to match Eq. (13a) and Eq. (13b) with their corresponding expectation values calculated at equilibrium. They are [1, 2],

$$T^{\mu\nu} = \varepsilon\, u^\mu u^\nu + p(\varepsilon, n)\, \Delta^{\mu\nu}, \tag{14a}$$

$$J^\mu = n\, u^\mu, \tag{14b}$$

where ε is the energy density, $p(\varepsilon, n)$ is the pressure density, and n is the charge density. Eqs. (14) exactly match Eqs. (13), upon the identifications $A \to \varepsilon$, $B \to p(\varepsilon, n)$, $C \to n$. Now, $p(\varepsilon, n)$ has a

[11] As is clear from the customary usage of the word dissipative, all orders higher than 0th order hydrodynamics are dissipative.

[12] Incidentally, this is why one can derive the linear response correlation functions from 1st order hydrodynamics [1]. And here comes another remarkable remark. Turning that on its head, having calculated a correlation function from a microscopic theory, from it one can extract information about all-order hydrodynamics [18].

[13] See, for example, [19, Section 13.8.3], for a discussion of how to take the nonrelativistic limit of the energy-momentum tensor, and consequently, the nonrelativistic limit of the vector current.

name; it is called the equation of state, which has to be taken as an input (see Section 3.3). Using the definition of the projector, Eq. (11), we can rewrite the energy-momentum in a standard form. For completeness, we rewrite both equations,[14]

$$T^{\mu\nu} = (\varepsilon + p)\, u^\mu u^\nu + p\, g^{\mu\nu}, \tag{15a}$$

$$J^\mu = n\, u^\mu. \tag{15b}$$

But where is the novelty, one might ask: we simply inserted these same equations, Eqs. (14), from thermodynamics? Indeed, hydrodynamics assumes having input from thermodynamics. But thermodynamics does not know about spacetime dependence. Because all the quantities in the constitutive relations were functions of the hydrodynamic variables, they were effectively functions of spacetime. In other words, in these last equations, we identify $\epsilon = \epsilon(x)$ as the local energy density, $p = p(x)$ as the local pressure density, and $n = n(x)$ as the local charge density; we are considering local thermal equilibrium. We have promoted them into slowly varying functions, which agrees with what we said in previous sections.

Finally, inserting Eqs. (15) in the conservation laws, Eqs. (1) and (2), gives us the equations of motion for hydrodynamics for a normal relativistic fluid at 0th order in derivatives. We shall not write (or solve) the equations explicitly. They can be found in [1, 2].

3. More modern formulation of (relativistic) hydrodynamics

As mentioned in Section 2, the underlying microscopic theory of any emergent behaviour is presumably a quantum field theory, as is the standard model of particle physics [22]. There are quantum field theories which share the property that short distances decouple from larger ones [23]. More precisely, the high-energy (UV) variables decouple from IR variables. After integrating out UV variables, we are left with IR variables only, which is dubbed as low-energy physics [3]. Effective field theories are the tools developed to consider such a decoupling. Incidentally, this was given a systematic explanation via the Wilsonian renormalization group [24, 25]. Given that hydrodynamics is an effective theory, one should, in principle, be able to derive it from the underlying quantum field theory [26]. Note that this would likely amount to deriving thermodynamic laws from the underlying quantum field theory along the way. Indeed [3, Section IIIG] claims that the a generalized version of the 2nd law of thermodynamics is derived from a general effective field theory. As should now be clear, having an action principle approach to hydrodynamics would be extremely beneficial, since already-developed techniques could be employed [26]. Dissipative (out of equilibrium, higher order) hydrodynamics, and other phenomena, from such an effective field theory point of view are reviewed in [3].

In this section, instead, we concern ourselves with ideal hydrodynamics from an action principle as covered in [27, Sections 5.1 and 5.2], [28], [29, Section II], [30, Section IIA], [31, Section IV]. In the spirit of effective field theories, we identify the symmetries for the case at hand, and then consider the most general action consistent with those symmetries. The shift in the point of view here is to consider the action principle description as the definition of ideal hydrodynamics, in the sense that we let the action principle structure tell us about how a perfect fluid should behave. Albeit, we do not get in all such details and subtleties which can be found in the references cited above. However, we derive the energy-momentum tensor (Section 3.1), identify the thermodynamic quantities (Section 3.2), and see how an equation of state restricts our action (Section 3.3).

[14] While it is still there, we omit writing the (ϵ, n) dependence on $p(\varepsilon, n)$ for aesthetic reasons.

3.1 Symmetries, the action, and the energy-momentum tensor

Now we consider a normal relativistic fluid without a conserved current. Let us pick our degrees of freedom that will appear in the action. We pick the IR degrees of freedom to be three scalar fields as functions of spacetime,[15]

$$\phi^I = \phi^I(x), \qquad I = 1, 2, 3. \tag{16}$$

Hence, the action is $S = S[\phi^I]$. This is an obvious upgrade—we are now looking for a field theory description.

To make contact with the fluid, we choose the scalar fields to be aligned with the space coordinates, when the fluid is in equilibrium (some specific time, say, $t = 0$) [28],

$$\phi^I(t = 0, x^i) = x^{i=I}, \qquad \text{or, more compactly,} \qquad \phi^I = x^I. \tag{17}$$

This enables us to specify the symmetries of ideal hydrodynamics, with hindsight. First, a perfect fluid is homogeneous. In other words, we require translational invariance, which in our degrees of freedom translates to,

$$\phi^I \to \phi^{I'} = \phi^I + a^I, \tag{18a}$$

where a^I = const. In addition, it is isotropic. In other words, we require rotational invariance, which translates to,[16]

$$\phi^I \to \phi^{I'} = R^I{}_J\, \phi^J, \tag{18b}$$

where $R^I{}_J$ denotes a representation of the rotation group $SO(3)$. Finally, we want our fluid to be incompressible. This is perhaps a more nontrivial symmetry. We require our fluid to be invariant under the group of volume preserving diffeomorphisms. Namely,

$$\phi^I \to \xi^I[\phi], \qquad \det \frac{\partial \xi^I}{\partial \phi^J} = 1. \tag{18c}$$

The denoted ξ^I is a different set of coordinates, but with the same volume element (the determinant condition).[17] Note that in light of the action, $S[\phi^I]$, all these symmetries are internal symmetries, as discussed in Section 2.1.

Now relaxing the equilibrium assumption in Eq. (17), the quantity which is invariant under symmetries given by Eqs. (18) is [27],

$$B = \det B^{IJ},$$

where

$$B^{IJ} = g^{\mu\nu}\, \partial_\mu \phi^I\, \partial_\nu \phi^J.$$

But any function of B is also invariant. Hence, we consider a generic function, $F(B)$. Note that derivatives of B such as the term $\partial_\mu B\, \partial^\mu B$ would also be invariant. However, such a term would effectively induce second derivatives on the fields. Since we are talking about ideal hydrodynamics,

[15] These fields are the so-called Lagrangian coordinates of the fluid [28]; for a discussion, see, for example, [32].

[16] Note that ϕ^I can be thought of as a 3-vector field under the rotation group $SO(3)$, which is an internal symmetry. However, it is still a scalar field under spacetime symmetries. Hence, by the usual convention in field theories, we refer to it as a scalar field.

[17] To elaborate a bit, for instance, the Einstein's equation is diffeomorphism invariant [13, pp. 434-435]. Diffeomorphisms can be considered as coordinate transformations, which amounts to saying that there is no preferred system of coordinates for the metric $g_{\mu\nu}$. Nevertheless, in our case, we are putting a further constraint: the determinant of the Jacobian J of the transformation has to be $\det J = 1$, which is just the determinant condition in Eq. (18c). Up to interpretation, this means we are considering a volume-preserving transformation which fits our requirement for the fluid to be incompressible. Intuitively, this goes back to the statement that one learns without knowing much about symmetry transformations: gasses are compressible, but liquids are not. Now perhaps we know a bit better: an ideal fluid is invariant under the group of volume-preserving diffeomorphisms.

we should only consider the lowest order derivative [28, Section II]. Consequently, because we require translational invariance given by Eq. (18a) and having no derivatives on the field would not work, the lowest order means having a first derivative on the fields, which is included in $F(B)$.

The claim is therefore that the most general action for a perfect fluid constrained only by the symmetries is,

$$S = \int \mathrm{d}^4x\sqrt{-g}\ \ F(B). \tag{19}$$

With the action at hand, we can derive the energy-momentum tensor $T_{\mu\nu}$, using the definition in Eq. (4) (see also [23, p. 418]). That procedure consists of finding the variation of the action with respect to the metric, which for Eq. (19) is,[18]

$$\delta S[g^{\mu\nu}] = \int \mathrm{d}^4x \left\{\delta\sqrt{-g}\cdot F(B) + \sqrt{-g}\cdot \delta F(B)\right\}. \tag{20}$$

Note that both g and B denote determinants.[19] In the first term in Eq. (20), we have the variation,

$$\delta\sqrt{-g} = -\frac{\sqrt{-g}}{2}\, g_{\mu\nu}\ \delta g^{\mu\nu}. \tag{21a}$$

And in the second term in Eq. (20), we have the variation,

$$\delta F(B) = F'(B)\ \delta B, \tag{21b}$$

but

$$\delta B = \delta \det B^{IJ} = B\ B_{IJ}\ \delta B^{IJ}, \tag{21c}$$

and finally,

$$\delta B^{IJ} = \partial_\mu\phi^I\ \partial_\nu\phi^J\ \delta g^{\mu\nu}. \tag{21d}$$

Collecting our results from Eqs. (21), inserting them where they came from and factorizing for convenience, for Eq. (20) we have,

$$\delta S[g^{\mu\nu}] = \int \mathrm{d}^4x\sqrt{-g}\ \ \delta g^{\mu\nu}\left\{-\frac{1}{2}\, g_{\mu\nu}\, F(B) + F'(B)\ B\ B_{IJ}\ \partial_\mu\phi^I\ \partial_\nu\phi^J\right\}. \tag{22}$$

As promised, using Eq. (22) in Eq. (4), we finally have the energy-momentum tensor,

$$T_{\mu\nu} = F(B)\ g_{\mu\nu} - 2F'(B)\ B\ B_{IJ}\ \partial_\mu\phi^I\ \partial_\nu\phi^J. \tag{23}$$

Now we have to match this to the energy-momentum for the perfect fluid, in order to identify the thermodynamic variables.

[18] Note that we vary with respect to the inverse metric, $g^{\mu\nu}$, which is consistent with the definition of the energy-momentum tensor. Defining the energy-momentum tensor with respect to the metric $g_{\mu\nu}$, gives an additional minus sign in the first term, which then does not agree the definition of the energy density ε as a Legendre transform of the Lagrangian density (see below).

[19] Hence, in both cases, we make use of the identity [13, p. 162],

$$\delta\ \det M = \det M\ \operatorname{tr}\!\left(M^{-1}\delta M\right),$$

which holds for a matrix M.

3.2 Thermodynamic quantities

The first step is to define [29],

$$B_{IJ}\,\partial_\mu\phi^I\,\partial_\nu\phi^J = u_\mu u_\nu + g_{\mu\nu},$$

which sits in the second term of our energy-momentum tensor, Eq. (23). Rewriting Eq. (23) by factorizing our covariant variables, we finally have,

$$T_{\mu\nu} = \left[F(B) - 2F'(B)\,B\right] g_{\mu\nu} + \left[-2F'(B)\,B\right] u_\mu u_\nu. \tag{24}$$

Now we can match this to the energy-momentum tensor for the perfect fluid given by Eq. (15a). What sits in front of $g_{\mu\nu}$ is the pressure density p, so,

$$p = F(B) - 2F'(B)\,B, \tag{25}$$

and what sits in front of $u_\mu u_\nu$ is $(\varepsilon + p)$, so with the help of Eq. (25), we get the energy density ε,

$$\varepsilon = -F(B). \tag{26}$$

To continue, we note the 1st law of thermodynamics in covariant form. In absence of chemical potentials, it has the form [17],

$$TS^\mu = pu^\mu - T^{\mu\nu}u_\nu, \tag{27}$$

where S^μ denotes the entropy current. For the energy-momentum tensor for a perfect fluid given by Eq. (15a), Eq. (27) becomes,

$$TS^\mu = (\varepsilon + p)\,u^\mu.$$

But also note that, for a perfect fluid at equilibrium, i.e. 0th order hydrodynamics, the entropy current is [1],

$$S^\mu = s\,u^\mu,$$

where s is the entropy density. Inserting the two results above in Eq. (27), we get the usual 1st law of thermodynamics,

$$T\,s = \varepsilon + p. \tag{28}$$

Using Eqs. (25) and (26), Eq. (28) gives us,

$$T\,s = -F'(B)\,B, \tag{29}$$

which does not uniquely define the temperature T and the entropy density s, but only their product. However, we also need to satisfy the 2nd law of thermodynamics. In covariant form, it reads

$$\partial_\mu S^\mu \geq 0, \tag{30}$$

where the equality sign holds for hydrodynamics without dissipation, i.e. a perfect fluid. Now since in [28] they show that there is an identically conserved current by construction,

$$\partial_\mu K^\mu = \partial_\mu(\sqrt{B}u^\mu) = 0,$$

they identify the entropy density to be,

$$s = \sqrt{B}, \tag{31}$$

which is consistent with the 2nd law of thermodynamics just stated in Eq. (30). Then, Eq. (29) and Eq. (31) give us the temperature T,

$$T = -F'(B)\,\sqrt{B}. \tag{32}$$

Finally, we have all the thermodynamic variables expressed in terms B. Effectively, therefore, all the thermodynamic quantities are expressed in terms of the scalar fields introduced in Eq. (16), which constituted our field theory.

3.3 Equation of state

As mentioned in Section 2.3, the pressure as a function of energy density, $p = p(\varepsilon)$, denotes the equation of state (now we do not have the charge density n). This relation is not derived from, but rather has to be brought into, hydrodynamics. More precisely, it has to be calculated from the underlying microscopic theory. In our case, having the equation of state, means that we get to fix $F(B)$. Let us take the case of ultra-relativistic matter, i.e. radiation. In that case, the equation of state reads [13, p. 35],

$$p = \frac{\varepsilon}{3}. \tag{33}$$

Incidentally, this is also the equation of state for the so-called conformal fluids [2, p. 26]. Taking the pressure p we obtained in Eq. (25) and the energy density ε we obtained in Eq. (26), the above Eq. (33) says that,

$$F(B) - 2F'(B)\,B = -\frac{F(B)}{3},$$

which is just a simple differential equation for $F(B)$. The solution is,

$$F(B) = c\,B^{2/3},$$

where c is a constant of integration. With the boundary condition $F(1) = -T_0^4$, our solution agrees with [30, Eq. (8)]. Note that T_0 is not the temperature which we have denoted by T in Eq. (32), but another quantity, as explained in Ref. [30]. The punchline is that given $F(B)$, all the thermodynamic quantities in Section 3.2 are fixed.

As opposed to the systematic way of progressing in the derivative expansion given the 0th order hydrodynamics, from the above action principle formulation for ideal hydrodynamics it is not at all obvious how to obtain higher-order hydrodynamics. In fact, as hinted at the start of Section 3, deriving dissipative hydrodynamics needs a whole other mechanism in effective field theories. Ref. [3] and references therein provide treatments of that machinery.

4. Summary and conclusion

Phenomena described by hydrodynamics are pervasive in nature. Because there is such a rich set of phenomena, including them in a single framework is far from trivial. We have covered the formulation of hydrodynamics as a derivative expansion in terms of the hydrodynamic variables, which provides for a more systematic way of organizing out-of-equilibrium phenomena, where we have only covered 0th order hydrodynamics in more detail. However, such a formulation has its deficiencies. Mostly because it needs additional input from thermodynamics and the underlying microscopic theory. In the hope of overcoming these difficulties in light of effective field theories, another formulation of hydrodynamics has set foot: hydrodynamics as an effective field theory, of which we only cover the naïve version of formulating ideal hydrodynamics from an action principle. Even with both these paradigms in mind, there is a lot more to say about hydrodynamics. Most of the novel observations have only become apparent in the 21st century. We shall conclude by remarking that hydrodynamics as glimpsed over in this work continuous to be a robust field of research in present days.

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