CAN ONE HEAR THE SHAPE OF A DRUM?

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In the first part, the Helmholtz equation is presented and a simple proof of isospectrality between two polygonal billiards is provided. Next, the geometric properties of billiards that can be deduced from the eigenvalue spectrum are examined and the statistical behavior of the spectrum is explored. In the last part, we two experimental realizations of isospectral billiards are provided which give experimental confirmation of isospectrality.

ALI LAHKO SLIŠIMO OBLIKO BOBNA?

V prvem delu je predstavljena Helmholtzova enačba in preprost dokaz izospektralnosti dveh biljardov v obliki večkotnika. Naslednji del je posvečen geometrijskim lastnostim biljardov, ki so dobljene s pomočjo lastnih vrednosti, in raziskavi statističnega obnašanja spektra. V zadnjem delu sta predstavljeni dve eksperimentalni potrditvi izospektralnosti različnih biljardov.

1. Introduction

In an article in 1966 Mark Kac [1], a Polish-Jewish mathematician known for his work in probability theory [1], asked an intriguing question: can one hear the shape of a drum? [2] That is, if we know the frequencies with which the membrane of a drum oscillates, can we know what shape is the boundary of a drum? In the paper, we will show that this is not the case!

Firstly, we will provide a simple example of isospectral domains (i. e. domains with the same spectrum of the Laplacian) in two-dimensional Euclidean plane with a simple construction procedure. We will call such domains *billiards*.

In the more physics-focused part of the paper, we will focus on quantum billiards and explore the connection between the spectrum and chaos. We will also describe several interesting experiments which provide experimental confirmation that there indeed exist two-dimensional billiards of different shapes that have the same eigenvalue spectrum.

2. The Helmholtz equation and isospectrality

In this paper, we will be primarily concerned with the eigenvalues of the Laplacian. We are solving the following partial differential equation (also called the *Helmholtz equation*)

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) f + Ef = \Delta f + Ef = 0, \tag{1}$$

where f is some function on a domain (or a billiard) $D \subset \mathbb{R}^2$. The function f must be at least two times differentiable. Whenever we are solving such an equation, we have to specify boundary conditions. Throughout this paper, we will assume *Dirichlet boundary conditions*, i. e. the function f vanishes on the boundary $\partial\mathcal{D}$

$$f\big|_{\partial\mathcal{D}} = 0. \tag{2}$$

[1] He was originally from Krzemienec (then Poland, now Ukraine) and completed his Ph.D. at the University of Lwów, where he was a member of Lwów School of Mathematics, which included famous names such as Stanisław Ulam and Stefan Banach. After completing his Ph.D. he received a scholarship in New York City in 1938. He was able to stay in America, while his family perished tragically in the Holocaust. After the war he worked mainly at Cornell University.

When one makes suitable approximations, the equation (1) describes numerous physical systems: oscillations of a thin vibrating membrane, electromagnetic field components in a metallic cavity and a free motion of a quantum particle confined in a box (time-independent Schrödinger equation). The one-dimensional case reduces to the problem of vibrating strings which was solved already in the 18th century. In the case of an oscillating membrane, the boundary condition (2) translates to having a clamped membrane at the boundary of the drum where there are no oscillations; in the quantum case it means that the wave function vanishes on the boundary (so called *hard walls*). The reason for calling such domains billiards is the following: we imagine a Hamiltonian such that the potential inside the domain is zero and infinity otherwise. This ensures mirror-like reflections at the boundary.

It is known [3] that the equation (1) with boundary conditions (2) has an infinite, but countable number of solutions. We can therefore denote the solutions with $\{f_n\}_{n\in\mathbb{N}}$ and eigenvalues with $\{E_n\}_{n\in\mathbb{N}}$, where we can ensure that $0 < E_1 \leq E_2 \leq \ldots$.

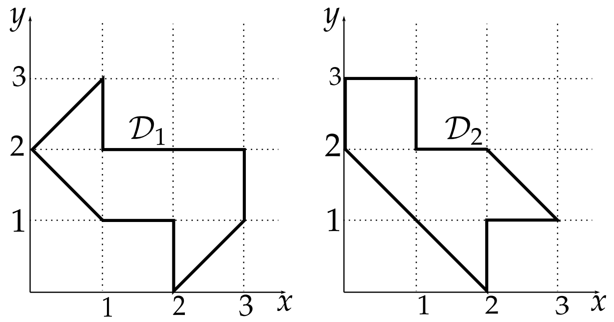

Figure 1. Isospectral billiards from [4].

We call two billiards *isospectral* if the sets of eigenvalues which are solutions to the equation (1) with boundary conditions (2) are the same. The answer 'no' to Kac's question would therefore require us to find two billiards of different shapes that are isospectral. In an article from 1992 [4] it was shown that such billiards in fact do exist and so one cannot hear the shape of a drum. An example of such a pair of isospectral billiards is depicted in Figure 1. The authors used advanced mathematical tools which far exceed the scope of this paper but in the next section, I will try to present a simple proof of isospectrality based on a pair of very simple polygonal billiards.

3. A simple proof of isospectrality

In this section, we will provide a simple proof of isospectrality of two simple billiards from Figure 2, which can then be generalized to other billiards – this method is called *paper folding proof.* It nicely captures the essence of isospectrality without being too mathematically involved [3].

We will illustrate the method with two simple billiards $\mathcal{B}_1$ and $\mathcal{B}_2$, presented in Figure 2. Both billiards are built out of 7 identical rectangular subdomains. Let us denote with ψ_1 the eigenfunction on $\mathcal{B}_1$ and E the corresponding eigenvalue. At the end of the proof we want an eigenfunction ψ_2 on the billiard $\mathcal{B}_2$ with the same eigenvalue E, which has to verify the Helmholtz equation (1), satisfy the boundary condition and have a continuous normal derivative inside $\mathcal{B}_2$.

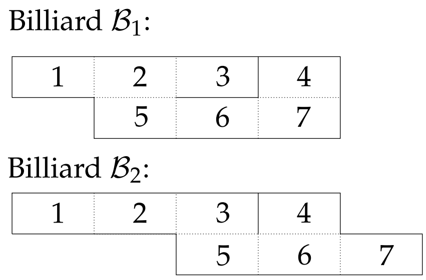

Figure 2. Billiards used in Section 3. Thick lines indicate Dirichlet boundary conditions, dotted lines are eye guides. Picture adapted from [3].

The main idea is to construct ψ_2 from a superposition of translations of ψ_1. Because equation 1 is linear the function ψ_2 will also be a solution. The only thing that needs to be checked are the boundary conditions – we are therefore looking for a superposition that satisfies Dirichlet boundary conditions.

We start by imagining three copies of $\mathcal{B}_1$ and fold them according to Figure 3. We now superimpose them onto one another as shown on the right of Figure 3 to get the shape of the billiard $\mathcal{B}_2$.

We make the following rules: stacking two rectangular subbilliards onto one another is equivalent to adding the functions defined on them; similarly, stacking the reverse of a subdomain is equivalent to subtracting them. The function ψ_2 on the billiard $\mathcal{B}_2$ (see Figure 2) is defined by these operations, e. g. on the 7th rectangular subdomain it is defined as

$$\psi_2\big|_{\text{subdomain 7}} = \psi_1\big|_{\text{subdomain 1}} - \psi_1\big|_{\text{subdomain 4}} + \psi_1\big|_{\text{subdomain 7}}. \tag{3}$$

Let's check that this procedure ensures that ψ_2 is zero on the boundary. If we look at the first rectangular subdomain in the billiard $\mathcal{B}_2$ we have

$$\psi_2\big|_{\text{subdomain 1}} = -\,\psi_1\big|_{\text{subdomain 1}} + \psi_1\big|_{\text{subdomain 2}} - \psi_1\big|_{\text{subdomain 5}}. \tag{4}$$

We know that $\psi_1|_{\text{subdomain 5}} = 0$ on the leftmost boundary (indicated by the thick line in $\mathcal{B}_1$). Because subdomains 1 and 2 are touching in $\mathcal{B}_1$ we know that $\psi_1\big|_{\text{subdomain 1}}$ and $\psi_1\big|_{\text{subdomain 2}}$ must be the same at that boundary and so

$$-\,\psi_1\big|_{\text{subdomain 1}} + \psi_1\big|_{\text{subdomain 2}} = 0. \tag{5}$$

But folded into the billiard $\mathcal{B}_2$ this is exactly the leftmost boundary of subdomain 1. Therefore ψ_2 is zero on the leftmost boundary of subdomain 1. This procedure is then repeated for every boundary of $\mathcal{B}_2$ for a full proof of isospectrality.

The procedure presented is clearly independent of the shapes of the subdomains which constitute the billiards, as all that is important is the way we 'glue' them together. For example, one can similarly construct isospectral billiards with a triangular subdomain. The pair of such billiards $\mathcal{D}_1$ and $\mathcal{D}_2$ is shown in Figure 1.

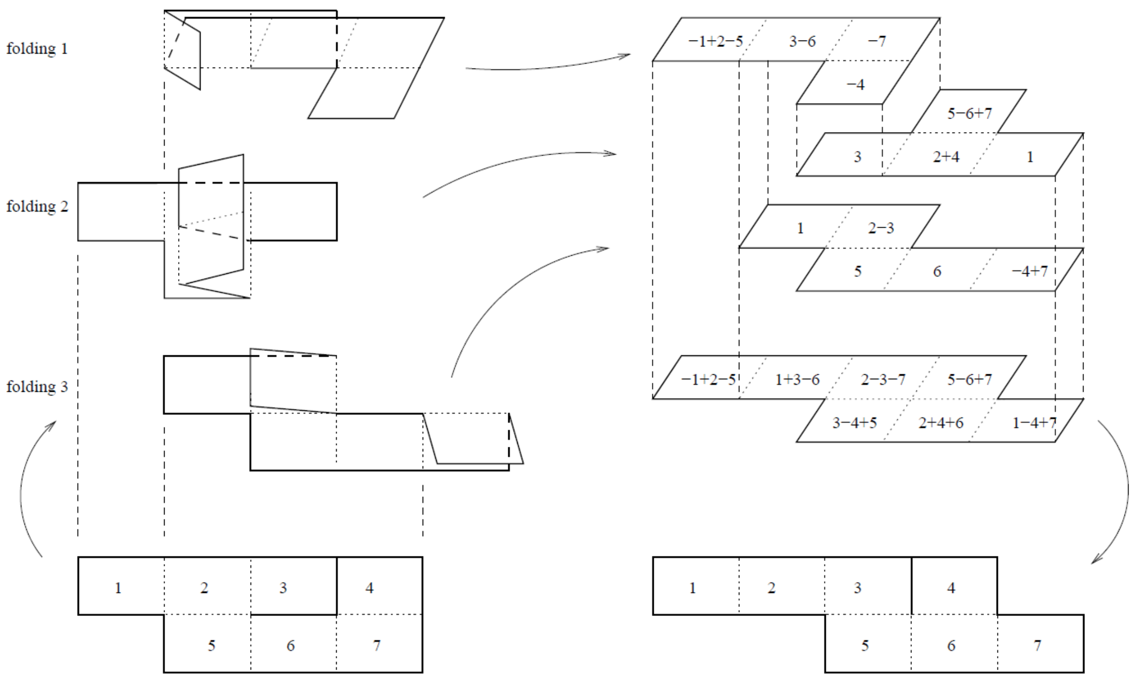

Figure 3. Paper folding procedure. Used with permission from [3].

4. What properties of drums can one hear?

Having seen in the previous section that one cannot hear the shape of a drum, the natural question arises: what geometric properties of the system can we extract from the eigenvalue spectrum? At the end of this section, we will also touch the topic of statistical distribution of eigenvalues and how it applies to two-dimensional billiards.

4.1 Mean density of eigenvalues

Given a billiard B we want to extract some information about the geometry of B based on the eigenvalue spectrum. We start by defining the density of energy levels

$$d(E) = \sum_n \delta(E - E_n) \tag{6}$$

and then integrating to get the counting function

$$\mathcal{N} = \sum_n \Theta(E - E_n) \tag{7}$$

which tells us how many eigenvalues are below energy E. Clearly, a pair of isospectral domains has the same counting function $\mathcal{N}(E)$.

We now want to examine the mean behavior of the counting function. Suppose we have an N-dimensional system in a domain $\mathcal{D}$ described by the Hamiltonian

$$H(\mathbf{p}, \mathbf{q}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{q}), \tag{8}$$

where

$$V(\mathbf{q}) = \begin{cases} 0, & \mathbf{q} \in \mathcal{D}, \\ \infty, & \text{otherwise.} \end{cases} \tag{9}$$

We use the well known *Thomas-Fermi approximation* which makes the assumption that each quantum state occupies a volume of $(2\pi\hbar)^N$ in phase space.

Applying this approximation to (7) we can, instead of an infinite sum, use the integral over phase space

$$\mathcal{N}(E) \approx \int \frac{\mathrm{d}^N p \mathrm{d}^N q}{(2\pi\hbar)^N} \Theta(E - H(\mathbf{p}, \mathbf{q})) = \frac{1}{\Gamma(N/2+1)} \left(\frac{m}{2\pi\hbar^2}\right)^{N/2} \int_{V(\mathbf{q})<E} \left[E - V(\mathbf{q})\right]^{N/2} \mathrm{d}^N q, \tag{10}$$

where in the last equality we integrated over the momentum part of the phase space [3]. When we describe the movement inside a N-dimensional domain of volume $\mathcal{V}$, we get

$$\mathcal{N}(E) \approx \frac{\mathcal{V}}{\Gamma(N/2+1)} \left(\frac{m}{2\pi\hbar^2}\right)^{N/2} E^{N/2}. \tag{11}$$

This is the first term in the *Weyl expansion*, first calculated by Hermann Weyl in 1911 [5].

For two-dimensional billiards, this reduces to (with the appropriate selection of units)

$$\mathcal{N}(E) \approx \frac{A}{4\pi} E, \tag{12}$$

where A is the area of the billiard [3]. One of the most striking consequences of equation 12 is the conclusion that if two billiards are to be isospectral they must have the same area. Or rather, formulated in the way of Mark Kac, one can hear the area of the drum.

Later several more terms were calculated and added to the Weyl expansion [3] for two-dimensional billiards

$$\mathcal{N}(E) \approx \frac{A}{4\pi} E \mp \frac{L}{4\pi}\sqrt{E} + K, \tag{13}$$

where L is the perimeter of the billiard, the $(+)$ is valid for Neumann boundary conditions, the $(-)$ sign is valid for Dirichlet boundary conditions and K contains information about the curvature of the boundary. More specifically, for billiards consisting of smooth edges γ_i which come together at corners of angles $0 < \alpha_i < 2\pi$ we get

$$K = \sum_i \frac{1}{24}\left(\frac{\pi}{\alpha_i} - \frac{\alpha_i}{\pi}\right) + \sum_i \int_{\gamma_i} \frac{\kappa(l)}{2\pi}\mathrm{d}l, \tag{14}$$

where $\kappa(l)$ is the curvature of the edges. As we can therefore see from equation (13), isospectral billiards must have the same area, perimeter and K, which in the case of polygonal billiards means

$$\sum_{\text{billiard 1}} \left(\frac{\pi}{\alpha_i} - \frac{\alpha_i}{\pi}\right) = \sum_{\text{billiard 2}} \left(\frac{\pi}{\alpha_i} - \frac{\alpha_i}{\pi}\right) \tag{15}$$

because curvature κ of straight lines is zero. In later sections we will also see that (13) has been checked experimentally.

4.2 Statistical behavior of eigenvalues

This subsection will deviate slightly from our central topic of isospectral billiards and focus on the statistical behavior of the eigenvalues of specific billiards. We start by considering a general Hamiltonian H which does not explicitly depend on time and by using the ansatz $\psi_n(x,t) = \psi_n(x)\exp(\frac{i}{\hbar}E_n t)$ arrive at the time-independent Schrödinger equation

$$H\psi_n(x) = E_n\psi_n(x), \tag{16}$$

we have found a constant of motion with a corresponding quantum number n. Without delving into the theory of chaotic systems let us mention that in completely chaotic systems, there are no other constants of motion [6]. If we expand ψ_n into a set of orthogonal functions $\phi_m(x)$

$$\psi_n(x) = \sum_m a_{nm}\phi_m(x), \tag{17}$$

we can also obtain the matrix representation of the Hamiltonian H_{nm} in the basis ϕ_m. If there exists a Hermitian operator R which commutes with H, we can use its eigenfunctions

$$R\phi_{n,\alpha} = r_n\phi_{n,\alpha} \tag{18}$$

to construct a block diagonal form of H with blocks H^n, which follows from

$$0 = \langle\phi_{n,\alpha}|RH - HR|\phi_{m,\beta}\rangle = (r_n - r_m)\langle\phi_{n,\alpha}|H|\phi_{m,\beta}\rangle \implies \langle\phi_{n,\alpha}|H|\phi_{m,\beta}\rangle = \delta_{n,m}H^n_{\alpha\beta}. \tag{19}$$

We can then use further symmetries to reduce H.

In the case of the hydrogen atom, we get four quantum numbers n, l, m and m_s. These represent the degrees of freedom of an electron, of which there are also four. We can now define what we mean by an integrable system: if the number of degrees of freedom is the same as the number of quantum numbers, we call the system *integrable* [6].

An example of a non-integrable system is an atomic nucleus. We are interested in the distribution of the energy levels of atomic nuclei which have been determined experimentally with nuclear spectroscopy. We define the spacing between energy levels as

$$s_n = E_n - E_{n-1} \tag{20}$$

and are interested in its distribution function $p(s)$. We present a data set known as the *nuclear data ensemble* which consists of results of the spectra for more than 30 nuclei (which have been normalized in order to be comparable) [6]. It can be shown [6] that the variable s obeys the Wigner-Dyson distribution

$$p(s) = \frac{\pi}{2} s \exp\left(-\frac{\pi}{4}s^2\right). \tag{21}$$

The interesting thing connecting this to two-dimensional billiards is the fact that the same distribution was observed experimentally in a quarter stadium billiard [6]. The experiment uses the same technique as the experiment presented in Section 5.1 (electromagnetic waves in a cavity). In the spirit of Kac, we could therefore say one cannot tell if one is 'listening' to a nucleus or a quarter shaped billiard as the spacing distributions of the spectrum 'sound' the same.

Finally, we consider an example of an integrable billiard, meaning that H is diagonal (in the appropriate basis) after all symmetries of the system have been considered. What kind of a spacing distribution do we expect in this case? We make the assumption that eigenvalues are uncorrelated based on the fact that each of them is in its own symmetry class. Then we can say that $p(s)\mathrm{d}s$ is the probability to find only one eigenvalue between s and $\mathrm{d}s$ from a given eigenvalue. We divide the interval of length s into N parts and we arrive at

$$p(s)\mathrm{d}s = \lim_{N\to\infty}\left(1-\frac{s}{N}\right)^N \mathrm{d}s, \tag{22}$$

where we multiplied the probability of finding no eigenvalues in any of the smaller N intervals and the probability of finding one between s and $s + \mathrm{d}s$. Here it was necessary that eigenvalues are uncorrelated. After taking N to infinity we get an exponential distribution

$$p(s) = \exp(-s). \tag{23}$$

We check this on a rectangular billiard with side lengths a and b. As the billiard is rectangular we can analytically get the following expression for the eigenvalues

$$E_n = \frac{\hbar^2 k_n^2}{2m_e} = \frac{\hbar^2}{2m_e}\left[\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi m}{b}\right)^2\right], \quad n,\ m \in \mathbb{N}. \tag{24}$$

It can be seen [6] that the distribution agrees with the Poisson distribution (23).

5. Experimental realization

In the following section, we will present experimental realizations of isospectral billiards $\mathcal{D}_1$ and $\mathcal{D}_2$ in Figure 1. Since in general there are no analytical solutions to equation (1) (only for specific set of boundary shapes), numerical and experimental calculation of eigenvalues is of even greater importance. It is also nice to see the correspondence between rather abstract mathematical formalism of isospectral theory and concrete, physical examples.

5.1 Electromagnetic waves in metallic cavities

The first experimental verification of isospectrality was done in 1994 by Sridhar [7], using microwaves in two thin copper cavities which were shaped as in Figure 1. As we will show in the following paragraphs, this can also be understood as a simulation of a quantum billiard.

From Maxwell's equation it follows [8, p. 356 - 360] that electromagnetic waves propagating through a hollow metallic cylinder (of arbitrary cross section and height h) obey the Helmholtz equations

$$\left(\Delta + k^2\right)\mathbf{E} = 0, \quad \left(\Delta + k^2\right)\mathbf{B} = 0, \tag{25}$$

where $k = \omega/c$ and ω is the angular frequency. The electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ have to satisfy the following boundary conditions

$$\hat{\mathbf{n}} \times \mathbf{E} = 0, \quad \hat{\mathbf{n}} \cdot \mathbf{B} = 0, \tag{26}$$

where $\hat{\mathbf{n}}$ is the normal vector. If the cylinder is sufficiently thin and all the walls parallel or perpendicular to $\hat{\mathbf{e}}_z$ then they are equivalent to

$$E_z\big|_{\partial D} = 0, \quad \nabla B_z\big|_{\partial D} = 0. \tag{27}$$

One of the solutions is a well known transverse magnetic mode (TM) which takes the form of [9]

$$E_z(x, y, z) = \psi(x, y) \cos\left(j\frac{\pi z}{h}\right), \quad j = 0, 1, 2, \ldots, \tag{28}$$

$$B_z(x, y, z) = 0, \tag{29}$$

where the scalar function ψ is the solution to the following Helmholtz equation

$$\left[\Delta + k^2 - \left(j\frac{\pi}{h}\right)^2\right]\psi = 0, \quad \psi\big|_{\partial D} = 0. \tag{30}$$

We can see that for $k < \pi/h$ only modes with $j = 0$ (because the constant in square brackets in equation (30) must be positive [8, p. 360]) are allowed and so we can write

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi_n = k_n^2\psi_n. \tag{31}$$

This is obviously the same equation as the stationary Schrödinger equation

$$-\frac{\hbar}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi_n = E_n\psi_n, \tag{32}$$

where quantum-mechanically the boundary conditions are those of an infinite potential well. This shows that for thin cavities where

$$h \leq \lambda_{\min}/2 = c/(2\nu_{\max}), \tag{33}$$

the quantum billiard obeys the same equation as electromagnetic waves which can therefore be used to simulate quantum billiards [9].

The experiment from Sridhar [7] was performed using two copper cavities in the shape of billiards in Figure 1; their height being $h = 6.3\,\mathrm{mm}$ and smaller side of the triangles (subdomains of the billiards) being $76\,\mathrm{mm}$ long. From (33) it follows that only frequencies below $\nu_0 = 25\,\mathrm{GHz}$ are allowed.

They obtained 54 lowest eigenvalues by determining the maxima of resonances in the transmission spectrum. In Figure 4 we can see the result of their experiment. It is obvious that both billiards $\mathcal{D}_1$ and $\mathcal{D}_2$ have the same resonant frequencies and are therefore isospectral. More precisely, they found relative discrepancies of $0.01\,\%$ to $0.2\,\%$ between pairs of billiard eigenvalues.

The results for the 54 eigenvalues also agree with Weyl's expansion (13) for the counting function $\mathcal{N}(E)$ with measured values for the area $A = 0.02\,\mathrm{m}^2$, perimeter $L = 78\,\mathrm{cm}$ and $K = 5/12$. This means that no eigenvalues were missed at these low frequencies.

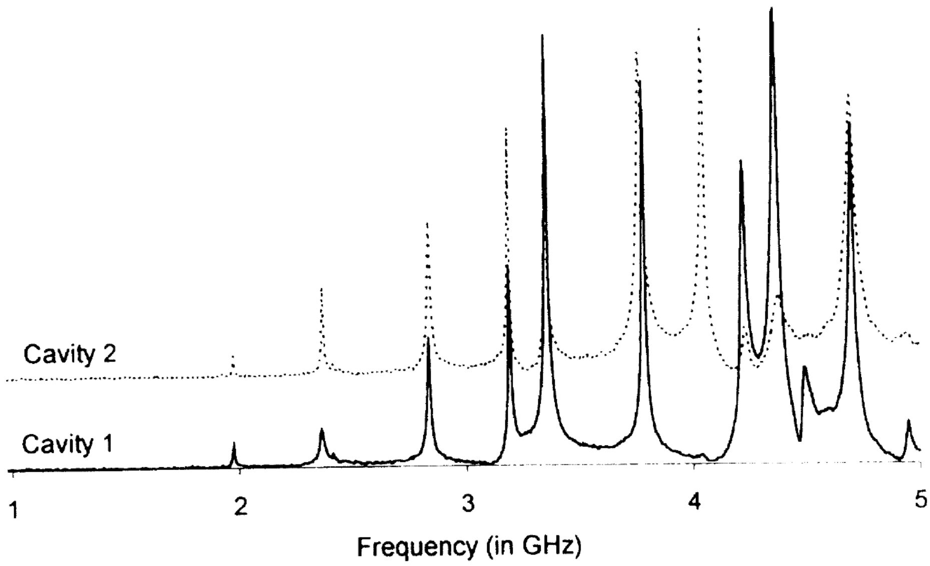

Figure 4. Transmission spectra vs frequency for the billiards $\mathcal{D}_1$ and $\mathcal{D}_2$ (shifted vertically for clarity). Isospectrality is obvious from the same resonant frequencies. It should be noted that the amplitude of the resonance is irrelevant, since it is dependent in the large part on the field strength at the probe location. This is especially noticeable with the 7th eigenvalue around 4 GHz which was measured at $\nu_7^{(1)} = 4.07030\,\mathrm{GHz}$ for the first billiard $\mathcal{D}_1$ and at $\nu_7^{(2)} = 4.07028\,\mathrm{GHz}$ for the second billiard $\mathcal{D}_2$ [7]. The width of the resonances is due to assembly of the cavities. Used with permission from [7].

5.2 Vibrations of a smectic liquid crystal

Another experimental proof of isospectrality was made using liquid crystals in a smectic phase – in this phase the molecules are arranged in layers [10]. The authors modeled two-dimensional drums using smectic liquid crystals whose transverse vibrations in vacuum obey the Helmholtz equation (1) with Dirichlet boundary conditions (2). A thin film of this liquid crystal is analogous to a membrane of a conventional drum. They have not only checked the spectrum but also the eigenfunctions of the two shapes in Figure 1.

First, they made the shapes in Figure 1 from stainless-steel with thickness 125 µm with the deviation from the desired boundary of about ± 5 µm. Then they applied the film of a smectic liquid crystal onto the shape, which after a few hours reaches equilibrium uniform thickness e of order 100 nm. Compared to the lateral dimension of the shape which was about 1 cm the two-dimensional approximation of the membrane is well-justified. The liquid crystal used is in smectic phase at room temperature. One of the advantages of using a liquid crystal in a smectic phase is that it possesses isotropic and uniform intrinsic tension γ (in this experiment $\gamma \approx 5 \times 10^{-2}\,\mathrm{N/m}$). The experimental set up is shown in Figure 5.

We can describe the transverse vibrations of the liquid crystals with the following wave equation

$$\gamma \Delta \psi = \rho e \frac{\partial^2 \psi}{\partial t^2}\,, \tag{34}$$

where γ is the intrinsic tension, ρ is the density and the function ψ describes vertical displacement from the equilibrium position with the boundary condition $\psi = 0$ for every point on the boundary. Using the product ansatz $\psi = z(x, y)T(t)$, we get the Helmholtz equation (1) for the function z.

To excite the film of the liquid crystal, they used a pinpoint electrode under the film to which voltage was applied. The position of the electrode was precise within 10^{-2} mm. To detect vibrations of the film they used the reflection of a laser on the surface of the liquid crystal; a photodiode is then used to detect the deviation of the laser from its initial direction. Using excitation frequencies

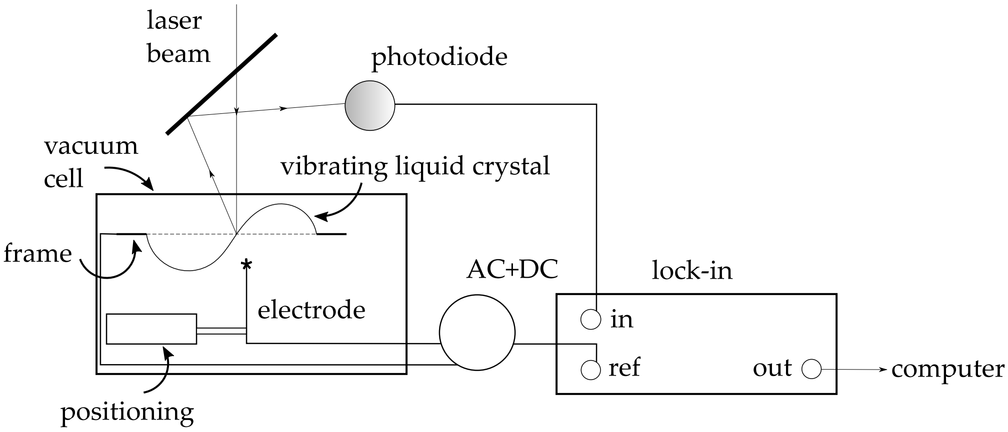

Figure 5. Experimental set-up for the vibrating liquid crystal. With the help of the photodiode we can measure the deviation of the laser beam from the starting direction. Figure reproduced from [10].

from 100 Hz to 1 kHz they were able to detect eigenvalues as resonance peaks. Moving the electrode allowed them to reconstruct eigenfunctions.

Their measured frequencies between billiards $\mathcal{D}_1$ and $\mathcal{D}_2$ differ by less than 0.15 % on average and by 0.34 % at maximum, which is well within their stated experimental relative error of order 0.5 % [10]. We can therefore conclude that the spectrum of both billiards is the same and that they have successfully answered 'no' to Kac's question.

6. Conclusion

We have explored the topic of eigenvalues of the Laplacian in two-dimensional billiards and with a simple example of an isospectral pair showed that one cannot hear the shape of a drum. We have then explored what possible geometric information one can get from the eigenvalue spectrum and also touched on the topic of statistical behavior of the spectrum of two-dimensional billiards. In the last section we have seen the connection between experiment and theory, where experimental realizations of isospectral billiards beautifully confirm the abstract mathematical theory.

For me, one of the most fascinating things about this topic is precisely the interplay between abstract mathematics and physics. Such a simple question of hearing shapes of drums which can almost be understood by a child, has produced immense results in abstract mathematics and then found its way back to physics, experimental physics even.

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