TESTI SPLOŠNE TEORIJE RELATIVNOSTI

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Članek predstavi štiri teste splošne relativnosti. Prvi trije so znani tudi kot klasični testi; gravitacijski rdeči premik, precesija Merkurjevega perihelija in ukrivljanje svetlobe zaradi Sončevega gravitacijskega privlaka. Predlagal jih je že sam Albert Einstein. Četrti test, ki je bil prav tako predlagan že dolgo časa, je bil nedavno uspešno izveden zaradi razvoja detektorjev gravitacijskih valov. Predstavljena je metodologija in fizika, na kateri so bili ti testi osnovani. Ta prispevek se v glavnem osredotoča na štiri eksperimente, medtem ko jih je bilo izvedeno še mnogo več, saj je splošna relativnost še vedno zelo bogata z eksperimentalnimi napovedmi.

TESTS OF GENERAL RELATIVITY

The article gives an overview of four tests of general relativity. The first three are known as the classic tests; prediction of gravitational redshift, perihelion precession of Mercury and Sun's deflection of light via gravity. They were already introduced by Albert Einstein himself. The fourth test, while also predicted a long time ago, has been recently verified due to the development of gravitational wave detectors. A description of the physics behind these experiments and the methodology used are presented. While this article focuses mainly on four, many other experiments and tests were conducted, as general relativity was and still is rich with experimental predictions.

1. Introduction

Upon its introduction in the early 20th century by Albert Einstein, general relativity (GR) quickly changed our understanding of gravity and the Universe in general. Many new processes were predicted from it, and experiments to prove their existence followed. At the time Einstein himself proposed three "classical tests" of his theory, he predicted the need for a gravitational redshift, explained the perihelion shift of Mercury and the deflection of light by the Sun. In this article, these three tests are described in some detail with the recently discovered gravitational waves at the end to demonstrate the predictive power contained within the theory of general relativity.
In order to discuss these experiments, we first need to understand some crucial concepts of GR. General relativity is a geometric theory, which is capable of describing physics in curved spacetime. This curved spacetime is mathematically described as a manifold, that being a topological space which, at any point, locally looks like a flat Euclidean space (we can imagine multiple small flat *tangent planes* that together compose a curved sheet). The mathematical definition is quite alienating, but the most basic example of a general manifold is the Earth's surface, a sphere. Now this concept introduces many problems for our common understanding of nature. Firstly, we no longer know how to compute the distance between two points. Similarly, we can no longer translate vectors in straight lines because we do not know what a "straight line" is. Therefore we need to introduce a concept or, better yet, a quantity that will describe the geometry of our manifold and give us the information as to how are local Euclidean spaces of different points sewn together into a single sheet. This quantity is called the *metric tensor* $g_{\mu\nu}$ (physical quantities are generally tensors in GR), and from it, we get a notion of "past" and "future", causality, and what the shortest distance is. We calculate the distance between two points in terms of the line element:

$$ds^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{1}$$

where the indices μ and ν describe the four coordinates of spacetime (one temporal and three spatial coordinates). Let us also mention that the coinciding superscript and subscript indices are summed over according to the *Einstein summation convention.* In this convention the metric tensor is also used for raising and lowering specific indices of vectors/tensors $g_{\alpha\beta}A^{\beta} = A_{\alpha}$. We can imagine dx^{μ} as the informal notion of an infinitesimal displacement (although actually it is derived as the rigorous notion of a basis one-form given by the gradient of a coordinate function [2]). It is also worth mentioning that the flat Minkowski metric used in this article uses the form $\eta_{\mu\nu} = diag(-1, +1, +1, +1)$. Another example of a metric tensor in the case of a sphere (two-dimensional curved plane) is of the form $g_{\mu\nu} = diag(1, sin^2\theta)$. We will also use the system of *natural units*, having units defined such that the numerical values of the selected physical constants (c) in terms of these units are exactly 1 (therefore we can omit them).

Now that we have a quantity that describes the geometry of our spacetime, we need to define a "connection", which gives us a way of relating vectors in the tangent spaces of nearby points by taking into account all the ways in which curvature manifests itself. We construct the connection from the metric in terms of the *Christoffel symbol*:

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_{\mu}g_{\nu\sigma} + \partial_{\nu}g_{\sigma\mu} - \partial_{\sigma}g_{\mu\nu}), \tag{2}$$

which looks like a tensor, but in fact does not behave like one. The fundamental use of a connection is to define a *covariant derivative* operator ∇ to perform the functions of the partial derivative, but in a way independent of local coordinates. The covariant divergence of a vector field V^{μ} is given by

$$\nabla_{\mu}V^{\mu} = \partial_{\mu}V^{\mu} + \Gamma^{\mu}{}_{\mu\nu}V^{\lambda}. \tag{3}$$

A connection also defines a specific way of keeping a tensor (physical quantity) "constant" along some path, meaning it is "parallel transported" (translations of vectors in Euclidean space). The crucial difference between flat and curved spaces is that, in a curved space, parallel transport will change the transported quantity and the result will depend on the path taken between the points. Therefore there is no *true* equivalent of parallel transport in curved spaces. We can however define *parallel transport* of a tensor to be the requirement that the covariant derivative of the tensor along the path vanishes. We can write the equation of parallel transport for a vector in the form:

$$\frac{d}{d\lambda}V^{m}u + \Gamma^{\mu}{}_{\sigma\rho}\frac{dx^{\sigma}}{d\lambda}V^{\rho} = 0. \tag{4}$$

With this, we can define a straight line as a path that parallel-transports its own tangent vector. We define a path $x^{\mu}(\lambda)$ with the help of an affine parameter λ, which is a well-defined measure of progression along this path. The tangent vector to the path is $dx^{\mu}/d\lambda$. We can now write the condition for a straight line in terms of the *directional covariant derivative*:

$$\frac{D}{d\lambda}\frac{dx^{\mu}}{d\lambda} = \left(\frac{dx^{\mu}}{d\lambda}\nabla_{\mu}\right)\frac{dx^{\mu}}{d\lambda} = 0, \tag{5}$$

this can be further rewritten in the *geodesic equation*:

$$\frac{d^2x^{\mu}}{d\lambda^2} + \Gamma^{\mu}{}_{\rho\sigma}\frac{dx^{\rho}}{d\lambda}\frac{dx^{\sigma}}{d\lambda} = 0, \tag{6}$$

where we can quickly check, that in the case of cartesian coordinates in Euclidean space ($\Gamma^{\mu}{}_{\rho\sigma} = 0$) the geodesic equation becomes $d^2x^{\mu}/d\lambda^2 = 0$, which is the equation for a straight line.

Finally, the technical expression of curvature is contained in the *Riemann tensor*. Everything we want to know about the curvature of a manifold is given to us by the *Riemann tensor*, which will vanish (i.e., it will be zero in all its components) if and only if the metric is perfectly flat. As we have mentioned true parallel transport is not possible on curved spaces and will always change the quantity transported. The Riemann tensor is defined exactly by the change of a vector/tensor that is transported around an infinitesimally small loop. As a matter of fact, the commutator of two covariant derivatives, measures the difference between parallel transporting the tensor first one way and the other, versus the opposite ordering:

$$\left[\nabla_\mu, \nabla_\nu\right] V^\rho = \nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho = R^\rho{}_{\sigma\mu\nu} V^\rho - T^\lambda{}_{\mu\nu} \nabla_\lambda V^\rho, \tag{7}$$

where $R^\rho{}_{\sigma\mu\nu}$ is the Riemann tensor and $T^\lambda{}_{\mu\nu}$ is the torsion tensor, which is 0 in GR and any theory which includes torsion is beyond the scope of this article. Using the previously defined covariant derivative in eq. (3), the Riemann tensor can be written as

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda} \Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda} \Gamma^\lambda{}_{\nu\sigma}. \tag{8}$$

While the Riemann tensor holds all information on the curvature of a manifold, it can be used to form two other measures of curvature. By contracting the first and third indices of $R^\rho{}_{\sigma\mu\nu}$, the symmetric *Ricci tensor* $R_{\mu\nu}$ and its trace, known as the *Ricci curvature scalar* R, can be formed:

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}, \qquad R = R^\mu{}_\mu = g^{\mu\nu} R_{\mu\nu}. \tag{9}$$

General relativity is a metric theory of gravity (not the only one), such theories are restricted by the existence of only one gravitational field that enters the equations of motion (therefore affecting matter), and that is the metric. Different metric theories are distinguished from one another only in the way matter (and possibly other additional fields) affects the metric.

In general relativity, we have the metric $g_{\mu\nu}$ and matter, which we describe as a tensor generalization of the mass density, known as the *energy-momentum tensor* $T_{\mu\nu}$. Conservation of energy is thereby conditioned with the equation $\nabla^\mu T_{\mu\nu} = 0$. Just as Maxwell's equations govern how the electric and magnetic fields respond to charges and currents, we now want to have field equations that govern how the metric responds to energy and momentum. From classical mechanics we know the Poisson equation for the Newtonian potential $\nabla^2 \Phi = 4\pi G_N \rho$, where G is the gravitational constant and ρ the mass density. Completely informally by analogy we expect that our metric will play the role of the gravitational potential, therefore we expect equations of a form $[\nabla^2 g]_{\mu\nu} \propto T_{\mu\nu}$. As it turns out we have a non zero quantity constructed from second derivatives: the Riemann tensor. By trying out different combinations and contractions of the Riemann tensor, and by satisfying its properties and the condition of energy conservation, we come to the correct expression. *Einsteins field equations* (EFE), which lie at the core of GR, are therefore:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{10}$$

where $R_{\mu\nu}$ and R are the Ricci tensor and scalar, $g_{\mu\nu}$ is the metric, $T_{\mu\nu}$ the energy-momentum tensor and G_N is Newton's gravitational constant (these field equations can be formally derived by starting with the *Hilbert action* and deriving the equations of motion).[2]

A solution to the EFE was provided by Karl Schwarzschild, which he derived while deployed to the Russian front in the first world war, this was the same year (1915) that general relativity was published. His metric can be shown in the form of a line element ds or proper time $d\tau$:

$$ds^2 = -d\tau^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \ , \tag{11}$$

where (t, r, θ, ϕ) are the usual spherical coordinates, M is mass, and G is Newton's gravitational constant. This Schwarzschild metric will often be referred to.

2. Gravitational redshift

The gravitational redshift of light was proposed by Albert Einstein, along with the precession of Mercury's orbit and the deflection of light by the Sun, as tests for his theory of gravity. At first it was believed to be a direct test of general relativity as its derivations often employ Schwarzschild solutions. Later it became clear that it is not a test for some theory of gravity in particular but of the underlying equivalence principle in general [1].

The *Einstein Equivalence Principle* (or EEP) is a generalization of something called a *Weak Equivalence Principle* (WEP), which states that in a small enough region of spacetime, the motion of a freely falling particle is the same in a gravitational field as it is in a uniformly accelerated frame. An example of WEP in Newtonian mechanics would be that the acceleration of an object is equal to the gradient of a gravitational potential, i.e., $a = -\nabla\Phi$.

Einstein made the leap from gravity being indistinguishable from uniform acceleration for the motion of freely-falling particles, to encompass any experiment. The EEP states that in a small enough region of spacetime the laws of physics reduce to special relativity and that it is impossible to detect the existence of a gravitational field with local experiments [2].

Gravitational redshift is often described in comparison with Doppler shift, and according to EEP, they are described in the same way. If one imagines two, for example, spaceships, one behind the other, traveling at some acceleration a. The first one then emits a signal, also known as a light wave, to the other. The second spaceship will then receive light with a slightly longer wavelength, it will be red-shifted and this effect is known as a Doppler shift.

According to EEP, this would be in effect the same if one of the ships was standing still on Earth at some elevation and the other at a slightly higher altitude also not moving. In this case we would call it gravitational redshift, and it can be computed from Doppler shift

$$\begin{aligned} &\text{Doppler:} && \frac{\Delta\lambda}{\lambda_E} = \frac{\Delta v}{c} = \frac{a\Delta y}{c^2}, \\ &\text{Gravity:} && \frac{\Delta\lambda}{\lambda_E} = \frac{a_G\Delta y}{c^2}, \end{aligned} \tag{12}$$

where λ_E is the emitted wavelength and $\Delta\lambda = |\lambda_E - \lambda_O|$ with λ_O being the observed wavelength. a_G would in this case be the gravitational acceleration (often referred to as g).

While the concept of gravity assisted redshift was introduced by Einstein, it would take until 1959 for it to be definitively proven with the Pound-Rebka experiment [8].

Essentially, it was a real-life version of the thought experiment described above for comparing Doppler and gravitational redshifts, using iron isotope ^{57}Fe, as its 14.4 keV nuclear transition has low levels of natural variation, as a source of photons and positioning it $\Delta y = 22.5$ meters below a detector. The configuration was also flipped to provide more data. However, the actual measurement was not as simple as it appeared, to stop γ photons from being absorbed on the way, the path was encased in a large Mylar cylinder and filled with liquid helium [10].

This was due to the required precision in red/blue -shift measurements, which was very high as the effect of redshift on Earth at a distance of ~ 22 meters, is incredibly small.

$$a_G = \frac{GM}{r^2} \quad \rightarrow \quad \frac{\Delta\lambda}{\lambda_E} = \frac{GM\Delta y}{r^2c^2} \tag{13}$$

By inserting known values in eq.(13), the obtained result is $\frac{\Delta\lambda}{\lambda_E} \simeq 10^{-15}$, which explains why a lot of liquid helium was needed.

A more formal derivation of gravitational waves with the use of General Relativity (GR) can be done in various ways; examples can be found in [1],[9], [7] and [6], with various degrees of complexity. A popular way to obtain the desired equations through GR is with the use of Schwarzschild metric.

With the assumption of static spacetime, we consider two observers equipped with ideal clocks at $x^{\mu}_{(1)}$ and $x^{\mu}_{(2)}$ (the numbers in brackets are for identification purposes). The coordinates are carefully chosen so that the their spatial part is constant. Then the first observer begins emitting light at the second. From the definition of proper time, we can write in terms of the first observer

$$d\tau^2 = g_{00}(x^{\mu}_{(1)}) \left(dx^0_{(1)}\right)^2 \ , \tag{14}$$

where $dx^0_{(1)}$ is the coordinate time and $d\tau$ will be the time between two wave crests (period of the emitted light) in terms of proper time. For the second observer, we write $\alpha d\tau$ the time between reception of the two emitted light waves in terms of proper time, as recorded by the second clock. In the same way as in eq. (14), we can write

$$\alpha d\tau^2 = g_{00}(x^{\mu}_{(2)}) \left(dx^0_{(2)}\right)^2 \ , \tag{15}$$

for the second observer. Here we can use the assumption of static spacetime, as it requires $dx^0_{(1)} = dx^0_{(2)}$, by using this relation in eq.14 and dividing eq.15 with it. We get

$$\alpha = \sqrt{\frac{g_{00}(x^{\mu}_{(1)})}{g_{00}(x^{\mu}_{(2)})}} \ , \tag{16}$$

where α can be expressed in terms of frequency as $\alpha = \nu_{(2)}/\nu_{(1)}$. We can also use the weak field approximation in the form

$$g_{00} \simeq 1 + 2\phi \ , \tag{17}$$

where ϕ is small. This can be applied in eq. (16) to get the following expression:

$$\alpha = \frac{\nu_{(2)}}{\nu_{(1)}} = \sqrt{\frac{g_{00}(x^{\mu}_{(1)})}{g_{00}(x^{\mu}_{(2)})}} \simeq \sqrt{\frac{1+2\phi_{(1)}}{1+2\phi_{(2)}}} \simeq \sqrt{\left(1+2\phi_{(1)}\right)\left(1-2\phi_{(2)}\right)} \simeq 1 + \left(\phi_{(1)} - \phi_{(2)}\right) \ , \tag{18}$$

where a Taylor series approximation was used by assuming small $\phi_{(1)}$ and $\phi_{(2)}$, additionally, a second-order term $\phi_{(1)}\phi_{(2)}$ was abandoned in the approximation. By reordering eq.(18), the suspiciously familiar equation appears

$$\frac{\nu_{(2)}}{\nu_{(1)}} \simeq 1 + \left(\phi_{(1)} - \phi_{(2)}\right) \quad \rightarrow \quad \frac{\Delta\nu}{\nu_{(1)}} \simeq \left(\phi_{(1)} - \phi_{(2)}\right) \quad \rightarrow \quad \boxed{\frac{\Delta\nu}{\nu_0} \simeq \frac{GM}{c^2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)} \ , \tag{19}$$

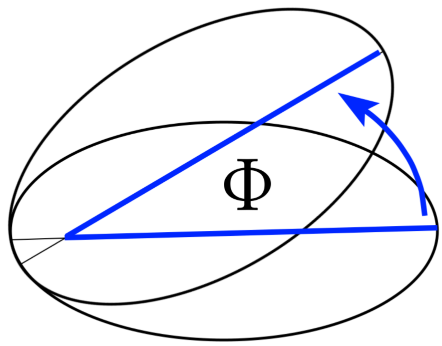
where $\phi = GM/r$ comes from the Schwarzschild solution and $\nu_{(1)}$ as the frequency of emitted light was renamed to ν_0. We can show that this is in agreement with the equivalence principle eq.(13):

$$\frac{\Delta\nu}{\nu_0} \simeq \frac{GM}{c^2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = \frac{GM}{c^2}\left(\frac{r_1 - r_2}{r_2 r_1}\right) \simeq \frac{GM}{c^2}\left(\frac{\Delta y}{r^2}\right) \ , \tag{20}$$

where we defined $\Delta y = |r_1 - r_2|$ and because r_1 and r_2 are both in reference to M, we approximated $r_1 r_2 \simeq r^2$.

3. Perihelion shift of Mercury

Observations of Mercury in the 19th century revealed an anomalous rate perihelion precession of the perihelion, and although various explanations were proposed, it remained unexplained until Einstein



Slika 1. A graphical representation of perihelion shift by ϕ, after some period of time.

resolved the issue with his general relativity.

Typically planets follow an elliptical orbit around the Sun, which is not fixed and can shift slightly with time in the form of perihelion precession. Perihelion is a point at which the planet is closest to the Sun, however its shift is measured as the rotation of the line connecting it to aphelion (the furthest point), as shown in figure 1.

The precession rate can be defined as the first derivative of ϕ and is typically measured in arcseconds per century. It can be written as

$$\dot{\phi}_{total} = \dot{\phi}_{GR} + \sum_{planets} \dot{\phi}_{planet} \qquad , \tag{21}$$

and its individual components can be compared. For example, the biggest effect on Mercury are other planets of our solar system, while GR contributes less than 10%, with the total value of about 43"/century. Table 1 lists the contributions for different planets.

Effect	**Value [arcsec/century]**	**Error [arcsec/century]**
Venus	**277.4176**	**<0.0001**
Earth/Moon	**90.8881**	**<0.0001**
Mars	**2.4814**	**<0.0001**
Jupiter	**153.9899**	**<0.0001**
Saturn	**7.3227**	**<0.0001**
General Relativity	**42.9799**	**0.0009**
Total	**575.3100**	**0.0015**

Tabela 1. Major contributions to the precession rate of the perihelion of Mercury (taken from [3]).

Before the GR contribution is calculated, it is worth revisiting classical Kepler motion and some of the equations that could be compared with their relativistic versions.

3.1 Classical elliptical orbits

The motion of planets, stars and other objects under the influence of gravity has and often still is calculated from Newton's laws [1], even after the introduction of general relativity their accuracy is still sufficient in most cases. These calculations typically apply Newton's second law and Newton's law of universal gravitation:

$$m\frac{d^2r}{dt^2} = -\frac{GmM}{r^2}\tilde{r} \tag{22}$$

where $\tilde{r} = \frac{\vec{r}}{|r|}$ is a unit vector, masses of the bodies involved m, M; and G is the Newtonian gravitational constant. To derive the relevant equations, it is useful to define angular momentum for a body with some mass m as:

$$\vec{L} = \vec{r} \times m\left(\frac{d\vec{r}}{dt}\right) = m\vec{h} \qquad \frac{d\vec{L}}{dt} = 0 \qquad \rightarrow \qquad \vec{h} = const. \tag{23}$$

Since in this case all orbits will be in the same plane and considering the conserved angular momentum, the equation of motion (22) can be rewritten to polar coordinates (R, ϕ), with $\tilde{\vec{R}}$ and $\tilde{\vec{\phi}}$ unit vectors.

$$\left[\frac{d^2R}{dt^2} - R\left(\frac{d\phi}{dt}\right)\right]\tilde{R} + \frac{1}{R}\frac{d}{dt}\left(R^2\frac{d\phi}{dt}\right)\tilde{\phi} = -\frac{GM}{R^2}\tilde{R} \quad \rightarrow \quad \ddot{R} - R\dot{\phi}^2 = -\frac{GM}{R^2} \quad \text{and} \quad \frac{d}{dt}\left(R^2\frac{d\phi}{dt}\right) = 0 \tag{24}$$

In polar coordinates, the equation for angular momentum (23) can be written as $L = mR^2\dot{\phi} = mh$, using this and by introducing a new variable $u = 1/R$ to eq.(24) a more simplified differential equation for $R = R(\phi)$ can be written.

$$\ddot{R} - \frac{h}{R} = -\frac{GM}{R^2} \quad \rightarrow \quad \boxed{\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2}} \tag{25}$$

Known also as the Binet equation (eq.(25)) it can be solved with

$$u = \frac{GM}{h^2} + C\cos(\phi - \phi_0) \quad \rightarrow \quad \boxed{\frac{1}{R(\phi)} = \frac{GM}{h^2}\left[1 + e\cos(\phi)\right]}, \tag{26}$$

where C and ϕ_0 are constants from integration and e is the orbital eccentricity, defined as $e = \frac{Ch^2}{GM}$.

3.2 General relativity and elliptical orbits

The approach in finding eq.(26) from general relativity, is to study timelike geodesics with Schwarzschild metric. Similarly, as in the Newtonian derivation, we assume the entire motion will be in a plane, therefore it is convenient to employ spherical coordinates with $\theta = \pi/2$. The argument to justify this choice is that if the initial position and a tangent vector of a geodesic lie in a $\theta = \pi/2$ plane, then the entire geodesic must be in this plane.
A timelike tangent of a curve, parameterized by proper time τ, can be written as

$$v^\mu = \frac{dx^\mu}{d\tau} = \dot{x}^\mu. \tag{27}$$

Because v^μ is timelike the following can be written using the Schwarzschild metric

$$g_{\mu\nu}v^\mu v^\nu = -\left[1 - \frac{2GM}{r}\right]\left(\frac{dt}{d\tau}\right)^2 + \left[1 - \frac{2GM}{r}\right]^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2 = -1\ , \tag{28}$$

whereas before G is the Newtonian gravitational constant and M is mass. By applying $\theta = \pi/2$ gets us

$$-\left[1 - \frac{2GM}{r}\right]\left(\frac{dt}{d\tau}\right)^2 + \left[1 - \frac{2GM}{r}\right]^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2 = -1\ . \tag{29}$$

With equation (29) and θ a constant, at least two more differential equations are required to compute (t, r, ϕ, θ). These can be obtained by using a Lagrangian $\mathcal{L} = g_{\mu\nu} v^\mu v^\nu$ and calculating the appropriate Euler-Lagrange equations: (eq.(30)).

$$\frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial v^\mu}\right) = 0 \ . \tag{30}$$

The Euler-Lagrange (EL) equations are the solution of a stationary action problem $\delta S = 0$, where the action is $S = \int \mathcal{L} d\tau$.
Computing equation (30) for $\mu = 0$ and $\mu = 3$ yields the following solutions

$$\begin{aligned} \mu = 0: &\quad \frac{d}{d\tau}\left[\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)\right] = 0 \quad \rightarrow \quad \boxed{\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right) = k} \\ \mu = 3: &\quad \frac{d}{d\tau}\left[r^2\left(\frac{d\phi}{d\tau}\right)\right] = 0 \quad \rightarrow \quad \boxed{r^2\left(\frac{d\phi}{d\tau}\right) = h} \end{aligned} \tag{31}$$

where k is a constant, and h is the specific angular momentum as defined in equation (23). One might notice that the Euler-Lagrange equation does technically provide all the differential equations needed for (t, r, ϕ, θ). The choice of $\mu = 0$ and $\mu = 3$ was deliberate; we do not require to use and compute eq.(30) with $x^\mu = r$ [6].
The next step is very similar to the classical derivation in the previous subsection; it is to introduce $u = 1/r$ again and to use equations (31) with equation (29).

$$\begin{aligned} -\left[1 - 2GMv\right]^{-1} k^2 + \left[1 - 2GMv\right]^{-1} h^2 \left(\frac{dv}{d\phi}\right)^2 + h^2 v^2 = -1 \\ k^2 - h^2\left(\frac{dv}{d\phi}\right)^2 - (1 - 2GMv)\, h^2 u^2 = 1 - 2GMv \\ \left(\frac{dv}{d\phi}\right)^2 - 3GMv^3 + v^2 - \frac{2GM}{h^2} = \frac{k^2 - 1}{h^2} \end{aligned} \tag{32}$$

This can then be derived by ϕ, as this will eliminate the constant term and the square of a derivative.

$$\frac{d^2 v}{d\phi^2} + v = \frac{GM}{h^2} \underbrace{+3GMv^2}_{} \tag{33}$$

The result (eq.(33)) is an equation that is suspiciously similar to the Newtonian eq.(25). The only difference is the last term with $3GMv^2$. We can treat this as a perturbation, as $3GMv^2$ is about $\sim 10^{-7}$ times smaller than $\frac{GM}{h^2}$ and the solution can then be in the form of $v = v_0 + v_1$.
The zeroth order solution v_0 is unsurprisingly the Newtonian equation (eq.(33)).

$$\frac{d^2 v_0}{d\phi^2} + v_0 = \frac{GM}{h^2} \quad \rightarrow \quad v_0 = \frac{GM}{h^2}\left(1 + e\cos(\phi)\right) \tag{34}$$

Eq.(34) can then be inserted into the first-order equation:

$$\begin{aligned} \frac{d^2 v_1}{d\phi^2} + v_1 = 3GMv_0^2 \quad &\rightarrow \quad \frac{d^2 v_1}{d\phi^2} + v_1 = \frac{3G^3M^3}{h^4}\left(1 + 2e\cos(\phi) + e^2\cos^2(\phi)\right) \\ &\rightarrow \quad \frac{d^2 v_1}{d\phi^2} + v_1 = \frac{3G^3M^3}{h^4}\left((1 + \frac{1}{2}e^2) + 2e\cos(\phi) + \frac{1}{2}e^2\cos(2\phi)\right) \ , \end{aligned} \tag{35}$$

where one can almost see the solution. By knowing that

$$\frac{d^2}{d\phi^2}(\phi\sin\phi) = 2\cos\phi \qquad \text{and} \qquad \frac{d^2}{d\phi^2}(\sin(2\phi)) = -3\cos 2\phi \ , \tag{36}$$

and using them with the eq. (35) one can determine the first order solution as

$$v_1 = \frac{3G^3M^3}{h^4}\left[\underbrace{e\phi\sin(\phi)} - \frac{1}{6}\cos(2\phi) - \left(1+\frac{1}{2}e^2\right)\right] . \tag{37}$$

where the only relevant term is $e\phi\sin(\phi)$. This is because if ϕ increases, only this term will increase v_1. The complete solution will then be

$$v \simeq \frac{GM}{h^2}\left[1 + e\cos(\phi) + \frac{3G^2M^2}{h^2}\phi\sin(\phi)\right] \quad \rightarrow \quad \boxed{v \simeq \frac{GM}{h^2}\Big[1 + e\cos\big((1-\varepsilon)\phi\big)\Big]} \tag{38}$$

where $\varepsilon = \frac{3G^2M^2}{h^2}$ and all the less-relevant terms from (37) are absent. From eq. (38) we can now see that ε represents the relative change in ϕ, which can be expressed as

$$\Delta\phi = 2\pi\varepsilon = \frac{6\pi G^2M^2}{c^2h^2} \simeq \frac{6\pi GM}{c^2(1-e^2)a} \tag{39}$$

where an approximation for the angular momentum $h^2 \simeq GM(1-e^2)a$ is used (a is the semi-major axis and e the orbital eccentricity). Inserting the known values for GM/c^2, a and e yields the result of 0.103" arcsec/orbit or 43 arcseconds per century [1], [6] and [2].

4. Deflection of light

4.1 Linearized gravity

The simplest way to examine the effect of the deflection of light is in terms of linearized gravity. In the Newtonian limit, we take a weak and also static gravitational field. While the weak field limit is sufficient for solar systems, we need to allow time dependence in order to examine fast-moving particles such as photons. We describe the metric as:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| << 1 \qquad \eta_{\mu\nu} = diag(-1,+1,+1,+1) . \tag{40}$$

From the metric, we can then derive Einstein's equations. We need to mention that the above metric does not completely specify the coordinate system on spacetime (there may be other coordinate systems in which the metric can still be written in this perturbative way but with a different perturbation). This *gauge invariance* indicates that physics will remain the same independent of the choice of coordinates (in electromagnetism, we can add any curl vanishing vector field to the vector potential, which is also a gauge symmetry). Lets here mention that this would mean GR is also a gauge theory, which it is as are all theories of fundamental interactions (in fact, it is often said that indeed general relativity is the unique theory of massless spin-2 particles at low energies (gravitons)).

Before tackling *gauge invariance* however, we can decompose the components of the metric perturbation by choosing a fixed inertial coordinate system in the Minkowski background spacetime and evaluating their transformation properties under spatial rotations. We decompose the tensor into individual pieces, which transform only into themselves (*irreducible representations* from group theory). The metric can be written as [2]:

$$\begin{gathered}\boxed{ds^2 = -(1+2\Phi)dt^2 + w_i(dtdx^i + dx^idt) + [(1-2\Psi)\delta_{ij} + 2s_{ij}]dx^idx^j} \\ h_{00} = -2\Phi \qquad h_{0i} = w_i \qquad h_{ij} = 2s_{ij} - 2\Psi\delta_{ij} \\ \Psi = -\frac{1}{6}\delta^{ij}h_{ij} \qquad s_{ij} = \frac{1}{2}\left(h_{ij} - \frac{1}{3}\delta^{kl}h_{kl}\delta_{ij}\right)\end{gathered} \tag{41}$$

From this metric, we can first derive the Christoffel symbols and from that the Riemann tensor, Ricci tensor, and the Ricci scalar. Because of *gauge invariance* ($h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$), we can also choose the *transverse gauge*: $\partial_i s^{ij} = 0, \partial_i w^i = 0$. The Einstein's equations, therefore reduce to the following set of equations:

$$\begin{aligned} G_{00} &= 2\nabla^2\Psi = 8\pi G T_{00} \\ G_{0j} &= -\frac{1}{2}\nabla^2 w_j + 2\partial_0\partial_j\Psi = 8\pi G T_{0j} \\ G_{ij} &= (\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi - \Psi) - \partial_0\partial_{(i}w_{j)} + 2\delta_{ij}\partial_0^2\Psi - \Box s_{ij} = 8\pi G T_{ij} \end{aligned} \tag{42}$$

4.2 Photon trajectory

Now in our case, we will model our static gravitating sources by dust (a good approximation for stars), a perfect fluid for which the pressure vanishes: $T_{\mu\nu} = \rho U_\mu U_\nu = diag(\rho, 0, 0, 0)$. For static sources we drop all time-derivatives and with the energy-momentum tensor obtain:

$$\begin{aligned} \nabla^2\Psi &= 4\pi G\rho \\ \nabla^2 w_j &= 0 \\ (\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi - \Psi) - \nabla^2 s_{ij} &= 0 \end{aligned} \tag{43}$$

We want only the fields that are **sourced** by the right-hand side to be nonvanishing. Therefore we quickly deduce that: $w^i = 0$. Similarly, from the trace in the third equation, we also see that: $2\nabla^2(\Phi - \Psi) = 0 \to \Phi = \Psi$, which changes the first equation into the classic Poisson equation. Finally, inserting $\Phi = \Psi$ into the remnants of the third equation gives: $\nabla^2 s_{ij} = 0$, the same reasoning as before implies $s_{ij} = 0$ for a well-behaved solution. We thereby get the following metric:

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)(dx^2+dy^2+dz^2) \qquad h_{\mu\nu} = -2\Phi diag(1,1,1,1) \qquad \nabla^2\Phi = 4\pi G\rho\ . \tag{44}$$

Now we consider the path of a massless particle (photon). We will solve the perturbed geodesic equation for a null trajectory $x^\mu(\lambda)$. Because of perturbation, we can write the geodesic as:

$$x^\mu(\lambda) = x^{(0)\mu}(\lambda) + x^{(1)\mu}(\lambda)\ . \tag{45}$$

Here $x^{(0)\mu}$ solves the geodesic equation in the flat background spacetime (straight null path). In the simplest case, we assume that the potential Φ does not drastically change along the background and true geodesics with the condition: $x^{(1)i}\partial_i\Phi << \Phi$. For convenience, we also define:

$$k^\mu \equiv \frac{dx^{(0)\mu}}{d\lambda}, \qquad l^\mu \equiv \frac{dx^{(1)\mu}}{d\lambda}\ . \tag{46}$$

First, we need to satisfy the condition for the null path: $g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$

$$\begin{aligned} &\text{0. order:} \quad \eta_{\mu\nu}k^\mu k^\nu = 0 \quad \to \quad (k^0)^2 = (k^i)^2 \equiv k^2 \\ &\text{1. order:} \quad 2\eta_{\mu\nu}k^\mu l^\nu + h_{\mu\nu}k^\mu k^\nu = 0 \quad \to \quad -kl^0 + l_i k^i = 2k^2\Phi \end{aligned} \tag{47}$$

We now turn to the perturbed geodesic equation $\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda}$. With the transverse gauge we get the Christoffel symbols:

$$\begin{aligned} \Gamma^0{}_{0i} = \Gamma^i{}_{00} &= \partial_i\Phi \\ \Gamma^i{}_{jk} &= \delta_{jk}\delta_i\Phi - \delta_{ik}\delta_j\Phi - \delta_{ij}\delta_k\Phi \end{aligned} \tag{48}$$

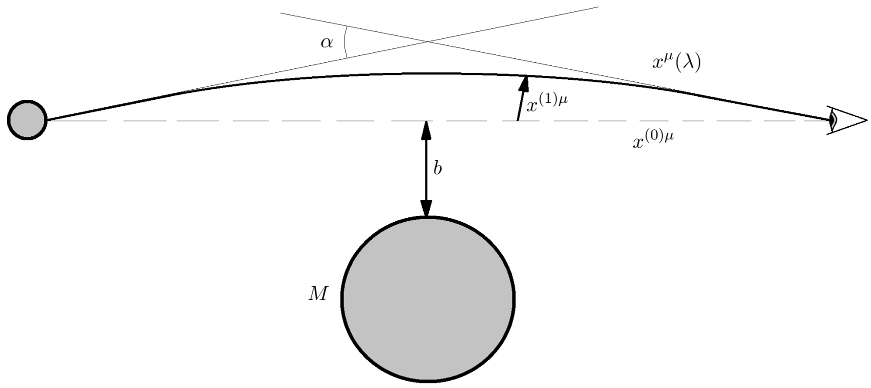
Since the Christoffel symbols are already first-order, the zeroth-order calculation of the geodesic just gives us a straight trajectory, while at first order we have:

$$\frac{dl^\mu}{d\lambda} = -\Gamma^\mu{}_{\rho\sigma}k^\rho k^\sigma \qquad \to \qquad \frac{dl^0}{d\lambda} = -2k(\vec{k}\cdot\vec{\nabla}\Phi), \qquad \frac{d\vec{l}}{d\lambda} = -2k^2\vec{\nabla}_\perp\Phi\ . \tag{49}$$

Here we introduced the *gradient transverse to the path*, defined as: $\vec{\nabla}_\perp \Phi = \vec{\nabla}\Phi - \vec{\nabla}_{||}\Phi = \vec{\nabla}\Phi - k^{-2}(\vec{k}\cdot\vec{\nabla}\Phi)\vec{k}$. Integrating for $\mu = 0$ gives:

$$l^0 = \int \frac{dl^0}{d\lambda} d\lambda = -2k \int (\vec{k}\cdot\vec{\nabla}\Phi) d\lambda = -2k \int (\frac{d\vec{x}}{d\lambda}\cdot\vec{\nabla}\Phi) d\lambda = -2k \int \vec{\nabla}\Phi \cdot d\vec{x} = -2k\Phi \; . \tag{50}$$

Plugging this back in (47) we get that the $\vec{l}$ and $\vec{k}$ are *orthogonal*: $\vec{l}\cdot\vec{k} = kl^0 + 2k^2\Phi = 0$. The



Slika 2. A deflected geodesic is decomposed into a background geodesic and a perturbation. The deflection angle represents the amount by which the wave vector rotates along the path.

value crucial for experimental validation will be the so-called **deflection angle** $\vec{\alpha}$. It represents the amount by which the original spatial wave vector is deflected as it travels from a source to the observer. It is a 2-dimensional value in the plane perpendicular to $\vec{k}$, the wave vector of the photon trajectory. We define the deflection angle with the help of the rotation of the wave vector, which can be calculated as follows:

$$\vec{\alpha} = -\frac{\Delta\vec{l}}{k}, \qquad \Delta\vec{l} = \int \frac{d\vec{l}}{d\lambda} b\lambda = -2k^2 \int \vec{\nabla}_\perp \Phi d\lambda \; . \tag{51}$$

Let us remember that λ represents a general affine parameter, we can define the actual physical spatial distance traversed as $s = k\lambda$, then we can finally write:

$$\vec{\alpha} = 2\int \vec{\nabla}_\perp \Phi ds \; . \tag{52}$$

Now we can finally evaluate the deflection angle in the case of a point mass, where we imagine the background path to be along the x-direction with an impact parameter defined by a transverse vector $b = |\vec{b}|$ pointing from the path to the mass at the point of closest approach. First, we write the potential and calculate its transverse gradient, after which we can integrate over it:

$$\begin{aligned} \Phi = -\frac{GM}{r} = -\frac{GM}{(b^2+x^2)^{1/2}} \\ \vec{\nabla}_\perp \Phi = \frac{GM}{(b^2+x^2)^{3/2}}\vec{b} \\ \alpha = |\vec{\alpha}| = 2GMb\int_{-\infty}^{\infty} \frac{dx}{(b^2+x^2)^{3/2}} = \frac{4GM}{b} \end{aligned} \tag{53}$$

The calculation was done in natural units $c = 1$. A factor of c^2 should be inserted in the denominator in other systems of units. The measurment of the deflection of light was historically the first time Einstein's theory correctly predicted a phenomenon that had not yet been detected. In 1919

Eddington [14] observed the positions of stars near the Sun during a total eclipse. The predicted effect is however, quite small. For our Sun we have $GM/c^2 = 1.48 \cdot 10^5 cm$, and the solar radius is $R = 6.96 \cdot 10^{10}$cm. This together leads to a maximum deflection angle of $\alpha = 1.75$ arcseconds.
In addition to the deflection of light, we can also observe a gravitational time delay. To the observer, the photons appear to slow down with respect to the background light cones. This was pointed out by Shapiro in 1964. It is worth mentioning that this time dilation is a different effect than the geometrical delay that arises because the photons travel a longer distance; in the case of our Sun, it is negligible [2].
Time dilation can be computed from the first-order perturbation of the trajectory:

$$t = \int \frac{dx^0}{d\lambda} d\lambda \qquad \delta t = \int \frac{dx^{(1)0}}{d\lambda} d\lambda = \int l^0 d\lambda = -2k \int \Phi d\lambda = -2 \int \Phi ds \ . \tag{54}$$

5. Gravitational waves

Gravitational waves generally arise from the weak-field equations with the absence of the energy-momentum tensor $T_{\mu\nu} = 0$. In the simplest case of linearized gravity [2], Einstein's equations reduce to the following conditions for well-behaved boundary conditions:

$$\begin{aligned} \nabla^2\Psi = 0 \qquad &\rightarrow \qquad \Psi = 0 \\ \nabla^2 w_j = 0 \qquad &\rightarrow \qquad w_j = 0 \\ (\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi) - \Box s_{ij} = 0 \qquad &\rightarrow \qquad \nabla^2\Phi = 0 \rightarrow \Phi = 0, \qquad \Box s_{ij} = 0 \end{aligned} \tag{55}$$

here we again chose the *transverse gauge* $(\partial_i w^i = 0, \partial_i s^{ij} = 0)$. Because the degrees of freedom (Φ, Ψ, w_i) are all but s_{ij} equal to zero, we call this the **transverse traceless gauge** $h^{TT}_{\mu\nu}$. The equations of motion become the following with a simple well-known solution in terms of Fourier modes:

$$\Box h^{TT}_{\mu\nu} = 0 \qquad \rightarrow \qquad h^{TT}_{\mu\nu} = C_{\mu\nu} e^{ik_\sigma x^\sigma} \tag{56}$$

$h^{TT}_{\mu\nu}$ must be spatial, traceless and transverse, therefore yielding:

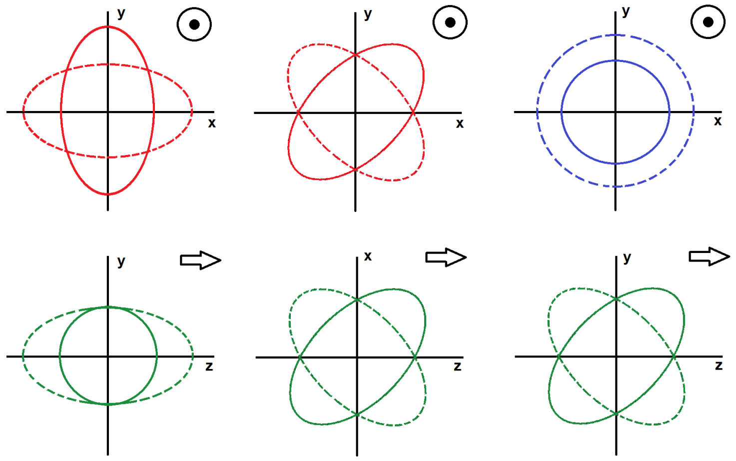
$$\begin{aligned} h^{TT}_{0\nu} = 0 \qquad &\rightarrow \qquad C_{0\nu} = 0 \\ \eta^{\mu\nu} h^{TT}_{\mu\nu} = 0 \qquad &\rightarrow \qquad \eta^{\mu\nu} C_{\mu\nu} = 0 \\ \partial_\mu h^{0\nu}_{TT} = 0 \qquad &\rightarrow \qquad iC^{\mu\nu} k_\mu e^{ik_\sigma x^\sigma} \rightarrow k_\mu C^{\mu\nu} = 0 \end{aligned} \tag{57}$$

We quickly see that our Fourier modes will have no temporal components and will be in the plane orthogonal to the wave vector $C^{\mu\nu} \perp k^\mu$. The traceless part only reduces the number of independent modes. The equations of motion yield an interesting result, that these waves will be null-like, which would mean they would travel at the speed of light:

$$0 = \Box h^{TT}_{\mu\nu} = \eta^{\rho\sigma}\partial_\mu\partial_\sigma h^{TT}_{\mu\nu} = \eta^{\rho\sigma}\partial_\mu(ik_\sigma h^{TT}_{\mu\nu}) = -\eta^{\rho\sigma}(k_\rho k_\sigma h^{TT}_{\mu\nu}) = -k_\sigma k^\sigma h^{TT}_{\mu\nu} \ . \tag{58}$$

Choosing a specific direction of the gravitational waves gives a further simplified form $k^\mu = (\omega, 0, 0, k^3) = (\omega, 0, 0, \omega)$. In general metric theories, we get six independent components, which indicate six different polarizations. Only three are transverse to the propagation as GR dictates (h_+, h_x, h_s), but even than our linearized gravity yields only two independent components [11]:

$$C^{jk} = \begin{pmatrix} h_s + h_+ & h_x & h_{v1} \\ h_x & h_s - h_+ & h_{v2} \\ h_{v1} & h_{v2} & h_L \end{pmatrix} \simeq \begin{pmatrix} h_+ & h_x & 0 \\ h_x & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \tag{59}$$



Slika 3. The six polarization modes for gravitational waves permitted in any metric theory of gravity. Shown is the displacement that each mode induces on a circular form of matter. The waves are set to propagate in the z-direction. There is no displacement out of the plane of the picture.

For experimental cross-examination it is particularly useful to analyze the *geodesic deviation equation*. Here we consider nearby particles with four-velocities described by a single vector field U^μ and a separation vector S^μ. Choosing a slow-moving particles $U^\mu = (1,0,0,0), \tau = x^0 = t$, we get a very useful equation:

$$\frac{D^2}{d\tau^2} S^\mu = R^\mu{}_{\nu\rho\sigma} U^\nu U^\rho S^\sigma \qquad \rightarrow \qquad \frac{\partial^2}{\partial t^2} S^\mu = \frac{1}{2} S^\sigma \frac{\partial^2}{\partial t^2} h^{TT}{}_\nu^{\ \mu} \,. \tag{60}$$

It is clearly evident that the separation vector will always be in the plane perpendicular to the four-velocity vector. It is pedagogically convenient to observe the effect of our gravitational waves modes separately as it indicates polarization of the waves in terms of geometrical deformations:

$$\begin{aligned} \frac{\partial^2}{\partial t^2} S^{1,2} &= \pm \frac{1}{2} S^{1,2} \frac{\partial^2}{\partial t^2} (h_+ e^{i k_\sigma x^\sigma}) \qquad \rightarrow \qquad S^{1,2}(t) = \left(1 \pm \frac{1}{2} h_+ e^{i k_\sigma x^\sigma}\right) S^{1,2}(0) \\ \frac{\partial^2}{\partial t^2} S^{1,2} &= \frac{1}{2} S^{1,2} \frac{\partial^2}{\partial t^2} (h_x e^{i k_\sigma x^\sigma}) \qquad \rightarrow \qquad S^{1,2}(t) = S^{1,2}(0) + \frac{1}{2} h_x e^{i k_\sigma x^\sigma} S^{2,1}(0) \end{aligned} \tag{61}$$

We can imagine the polarizations in terms of deforming a circular form of matter. While the first polarization does not show any coupling, the components of the separation vector will stretch and shrink out of phase orthogonally to each other. In contrast, the second polarization will yield deformations at a 45^o angle. Observing a more general metric would show even more modes or polarizations, which GR does not predict. Analyzing the exact polarizations of distant gravitational waves can therefore, serve as a direct tool for checking the validity of the theory. It is very interesting to note that while GR permits only the two modes h_x, h_+ these actually indicate massless spin-2 particles (gravitons).

5.1 Binary star

In practical experiments, we usually observe gravitational waves produced by a source, so we cannot expect $T_{\mu\nu} = 0$. It is a bit tedious but entirely possible to derive the existence of waves (with linearized gravity) from a **quadrupole moment tensor** $I_{ij}(t) = \int y^i y^j T^{00}(t, \vec{y}) d^3 y$. For easier derivation we define the *trace-reversed perturbation* $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}, \bar{h} = \eta^{\mu\nu} \bar{h}_{\mu\nu} = -h$. In vacuum this is the same: $\bar{h}^{TT}_{\mu\nu} = h^{TT}_{\mu\nu}$. We solve the equations of motion by using the retarded Green function (disturbances in the gravitational field are calculated in terms of events on the past light cone). With the help of Fourier transformations, we obtain the solution in terms of a **quadrupole formula** [11]:

$$\Box \bar{h}_{\mu\nu} = -18\pi G T_{\mu\nu} \qquad \rightarrow \qquad \bar{h}_{ij}(t,\vec{x}) = \frac{2G}{r}\frac{d^2 I_{ij}}{dt^2}(t-r) \ . \tag{62}$$

Dipole momentum does not contribute anything because oscillation of the center of mass of an isolated system violates the conservation of momentum. Gravitational radiation is therefore, much weaker than electromagnetic, since the quadrupole moment, which measures the shape of the system, is generally smaller than the dipole moment.
A typical case where such gravitational radiation is produced by the quadrupole moment is a binary star(two stars in orbit around each other). Let us say they are in circular orbit both with the same mass. We can write the explicit path of stars a and b, which give us the energy density:

$$\begin{aligned} & x^1_{a,b} = \pm R\cos\Omega t, x^2_{a,b} = \pm R\sin\Omega t \\ & \quad \rightarrow \quad T^{00}(t,\vec{x}) = M\delta(x^3)\left[\delta(x^1 - R\cos\Omega t)\delta(x^2 - R\sin\Omega t) + \delta(x^1 + R\cos\Omega t)\delta(x^2 + R\sin\Omega t)\right] \ . \end{aligned} \tag{63}$$

Integrating the quadrupole moment than gives us:

$$\bar{h}_{ij}(t,\vec{x}) = \frac{8GM}{r}\Omega^2 R^2 \begin{pmatrix} -\cos 2\Omega(t-r) & -\sin 2\Omega(t-r) & 0 \\ -\sin 2\Omega(t-r) & \cos 2\Omega(t-r) & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \tag{64}$$

5.2 Detecting gravitational waves

The most common sources of gravitational waves, that we expect to be measurable, are large or better yet heavy binary systems, that we already non-relativistically described in the previous part. By detecting gravitational waves, we mean observing the influence of the gravitational wave on test bodies. Such observatories are usually laser interferometers. The laser is sent on a beamsplitter and travels along two orthogonal paths of the same length bouncing multiple times front to end. A gravitational wave will cause a change in the length of the paths that will be related to the waves amplitude $\frac{\delta L}{L} \sim h$. We are then able to measure the phase shift of the splitted beam. The frequency of the orbit of a binary system and thus of the produced gravitational waves is given non-relativistically by $f = \Omega/(2\pi) \sim (cR_s^{1/2})/(10R^{3/2})$, here we use the Schwarzschild radius $R_s = 2GM/c^2$. We then estimate the amplitude as $h \sim (R_s^2)/(rR)$. Assuming a black-hole binary system with ten solar masses ($R_s \sim 10^6$cm), at a cosmological distance ($r \sim 100Mpc \sim 10^{26}$cm) and the radius of the orbit to be ten times the Schwarzschild radius ($R = 10R_s$), we get $f \sim 10^2 s^{-1}, h \sim 10^{-21}$ for an expected gravitational wave.

$$\delta L \sim 10^{-16}\left(\frac{h}{10^{-21}}\right)\left(\frac{L}{\mathtt{km}}\right)\mathtt{cm} \qquad \rightarrow \qquad \delta\phi \sim 200\left(\frac{2\pi}{\lambda}\right)\delta L \sim 10^{-9} \tag{65}$$

This would require a sensitivity of much less than the size of the constituents atoms out of which any conceivable test masses would have to be made. The accumulated phase shift of a splitted beam with $\lambda \sim 10^{-4}$cm making 100 round trips through the cavity arms would however, be detectable as estimated in the above expression[2].

5.3 Speed of gravity

According to general relativity, gravitational waves should propagate at the speed of the limit. That would hold at least in the limit at which the wavelength of gravitational waves is small compared to the curvature of the background spacetime. In other theories, this could differ due to additional coupling of gravitation to "background" fields. Another interesting perspective at the speed of gravitational waves would be in the sense of massive gravitons. In a local inertial frame we would write:

$$\frac{v_g^2}{c^2} = 1 - \frac{m_g^2 c^4}{E^2} \tag{66}$$

for a massive propagating particle, here v_g is the speed of gravity, while m_g and E are the graviton rest mass and energy, respectively [15]. While the speed of gravity could be measured from the Shapiro time delay, it contributes to a higher-order term and is therefore, difficult to measure. The most used method for measuring the speed of gravity is to compare the arrival times of a gravitational wave and electromagnetic wave from the same event, for example, a supernova or a prompt gamma-ray burst. For a source at a distance D, the resulting value of the discrepancy is:

$$1 - \frac{v_g}{c} = 5 \times 10^{-17} \left(\frac{200Mpc}{D}\right)\left(\frac{\Delta t}{1s}\right), \qquad \Delta t = \Delta t_a - (1+Z)\Delta t_e \tag{67}$$

$\Delta t_a, \Delta t_e$ are the differences in arrival time and emission time of the two signals, respectively. Z is the redshift of the source. In most cases though, Δt_e is based on observation or modeling and is not known exactly [15].

6. Conclusion

Within this article, we presented a brief overview and presentation of experiments crucial to the validity and analysis of Einstein's theory of gravity - general relativity. Proposed by Einstein himself, the classical tests are fundamental to the theory and show a key understanding of spacetime in general. Even though general relativity has passed this test successfully it is still of great importance to reexamine and verify them at higher accuracy. Measurements have so far been done only at specific scales and limits, such as the weak-field limit. With more accurate experiments it might be possible to discover very small deviations from what general relativity predicts and might indicate the validity of other metric theories of gravity. It could very well turn out that the correct equations should include two metrics, and the second metric could be relevant only at high energies; this could indicate that the speed of light might be energy-dependent. These theories are known as *bimetric gravity*.

As we have mentioned, the 21st century presents the start of a completely new frontier for observing and experimentally testing general relativity. The capability of detecting and measuring gravitational waves has opened up new ways to confirm already checked aspects of GR as well as to validate other not yet tested properties of GR. Searching and checking different polarization modes might indicate the validity of other theories of gravity since, as we mentioned, GR predicts only two modes that correlate with a spin-2 particle (graviton). In massless scalar-tensor gravity, a third polarization mode would add a spin-0 scalar field. In particular, observations of gravitational wave background with space-based interferometers might find scalar, vector, and tensor polarization modes. In addition, these background waves may have been produced a very long time ago and could hold information about the physics of the early Universe (beyond even the cosmic microwave background)[16].

A particularly intriguing concept of GR that has not yet been examined completely is the exact speed of gravity. While certain measurements have already been carried out, there is still much debate on the exact behaviour of gravitational waves. The most common measurements are done by observing binary neutron star mergers that emit both gravitational waves and short gamma-ray bursts. Comparing the detection of both types of radiation, that arise from the same event, yields stringent limits on the relative speed of gravity compared to the speed of light. The exact value is not possible to measure since it is difficult to even estimate the time delay between the emission time of gravitational waves and electromagnetic radiation. While such measurements are currently carried out at large observatories like LIGO and VIRGO, an experiment on much smaller scales has recenty been proposed. A setup composed of two colinear masses and a detector in between. Periodic oscillation of these two test bodies could cause constructive interference, through which the speed of gravity could be theoretically measured[12][17].

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