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The article presents Majorana modes in condensed matter physics. The Kitaev chain model with a $p-$wave superconductor pairing is approached and Majorana operators are introduced. Step by step topological protection, non-abelian statistics and braiding, key ingredients for topological quantum computation, are presented. The theoretical Kitaev model is modified to a realistic spinful fermion nanowire model with $s-$wave superconductor induced pairing and spin-orbit coupling. Lastly, experimental evidence is presented.

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V članku so predstavljeni Majorana fermioni v fiziki kondenzirane snovi. Predstavljen je Kitaev model verige s $p-$superprevodno sklopitvijo, temu pa sledi definicija Majorana operatorjev. V nadaljevanju so opisani: topološka zaščitenost stanj, neabelska statistika in pletenje (ang. "braiding"), ki so ključni sestavni deli topološkega kvantnega računalništva. Teoretični Kitaev model je nato preoblikovan do realističnega modela, ki upošteva spin fermionov, $s-$superprevodno sklopitev in spin-orbit interakcijo, nazadnje pa je predstavljen tudi eksperimentalni dokaz Majorana stanj.

1. Introduction

In 1937 *Ettore Majorana* suggested that electrically neutral particles, such as neutrons and neutrinos, could be, in the context of quantum field theory, represented by a single real field. While neutrons and antineutrons are distinct particles, it remains unknown whether or not neutrinos are Majorana particles and it is to be seen if Majorana fermions exist in nature as elementary particles. [1]

However, in condensed matter physics Majorana fermions could be realised as quasiparticles, excitations in solids, with interesting properties applicable in future quantum computation hardware.

In the article the Kitaev chain is introduced and topology in superconductors is briefly mentioned. Secondly, braiding inside a one-dimensional system and non-abelian statistics are presented, and lastly, a detection of Majorana fermions in condensed matter is discussed.

2. The Kitaev chain model

We start with a simple Hamiltonian, proposed by *A. Kitaev* [2], describing a finite 1D tight-binding chain model of length N with superconducting $p-$wave pairing consisting of chemical potential, a hopping term and a superconducting term:

$$H = -\mu \sum_{i=1}^{N} c_i^\dagger c_i - t \sum_{i=1}^{N-1} \left(c_{i+1}^\dagger c_i + \text{h.c.} \right) + \Delta \sum_{i=1}^{N-1} \left(c_i c_{i+1} + \text{h.c.} \right), \tag{1}$$

where μ is the chemical potential, t is the hopping parameter, Δ is the superconducting gap of a p-wave superconductor and as usual c_i, $c_i^\dagger$ are the fermion annihilation and creation operators, which obey the following anti-commutation relations: $\{c_i, c_j^\dagger\} = \delta_{i,j}$ and $\{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0$.

2.1 Majorana operators

Let us introduce Majorana operators and rewrite equation (1). Majorana operators γ_{2i-1} and γ_{2i} acting on site i are defined as the decomposition of c_i and $c_i^\dagger$ to their complex components:

$$c_i = \frac{1}{2} \left(\gamma_{2i-1} + i\gamma_{2i} \right), \tag{2}$$

$$c_i^\dagger = \frac{1}{2} \left(\gamma_{2i-1} - i\gamma_{2i} \right). \tag{3}$$

We invert equations (2) and (3) and express γ_{2i-1} and γ_{2i}:

$$\gamma_{2i-1} = c_i^\dagger + c_i, \tag{4}$$

$$\gamma_{2i} = i\left(c_i^\dagger - c_i\right). \tag{5}$$

From the last two equations it can be directly seen that γ_i are indeed Majorana operators as they are clearly hermitian $\gamma_i = \gamma_i^\dagger$. Majorana operators satisfy the anti-commutation relation:

$$\{\gamma_i, \gamma_j\} = 2\delta_{i,j}. \tag{6}$$

Now we are ready to rewrite the Hamiltonian (1) in terms of Majorana operators:

$$H = -\frac{i}{2}\sum_{i=1}^{N} \mu\gamma_{2i-1}\gamma_{2i} + \frac{i}{2}\sum_{i=1}^{N-1}\left(\left(t+\Delta\right)\gamma_{2i}\gamma_{2i+1} + \left(-t+\Delta\right)\gamma_{2i-1}\gamma_{2i+2}\right). \tag{7}$$

Consider two possible choices of parameters μ, t and Δ:

$$\mu < 0,\ t = \Delta = 0: \quad H = -i\frac{\mu}{2}\sum_{i=1}^{N}\gamma_{2i-1}\gamma_{2i}, \quad \text{and} \tag{8}$$

$$\mu = 0,\ t = \Delta > 0: \quad H = it\sum_{i=1}^{N-1}\gamma_{2i}\gamma_{2i+1}. \tag{9}$$

These two fully dimerized cases correspond to two distinct topological phases, where the first one is called *trivial*, and the second is *non-trivial* or *topological* (see sect. 2.2). This distinction comes from the fact that Majorana operators are coupled on the same site i (eq. 8), or neighbouring sites i and $i+1$ (eq. 9). The difference is pictorially presented in figure 1.

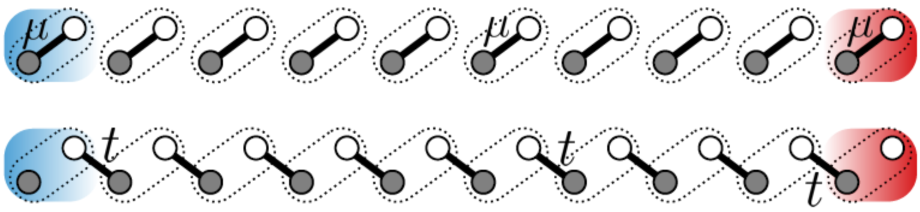


Figure 1. Two fully dimerized cases are shown corresponding to eqs. (8) (upper) and (9) (lower). The topological phase (lower) hosts on each edge an uncoupled Majorana operator. Adapted from [3].

An interesting property arises from eq. (9): the Hamiltonian does not include γ_1 and γ_{2N} operators, therefore $[H, \gamma_1] = 0$ and $[H, \gamma_{2N}] = 0$ commutation relations hold. Those two edge operators correspond to zero energy eigenstates, *Majorana zero modes* (MZMs). MZMs can only exist in pairs as they are "halves" of a fermion operator (eqs. 2 and 3).

For further investigation it is useful to calculate the bulk spectrum (here we consider the periodic boundary conditions, so no edge modes are presented), which is obtained from transformation to the momentum space. Without derivation we write the *Bogoliubov-de Gennes Hamiltonian* H_{BdG} in momentum space [4], which can be reduced to 2×2 matrix blocks $H(k)$, where k is a crystal momentum in the first Brillouin zone:

$$H_{\text{BdG}} = \sum_k H(k) c_k^\dagger c_k, \tag{10}$$

$$H(k) = \left(-2t\cos(k) - \mu\right)\tau_z + 2\Delta\sin(k)\tau_y, \tag{11}$$

where τ_i are the Pauli matrices acting in a particle-hole *Nambu space.* [5] From eq. (11) we derive the band structure of the Kitaev chain model:

$$E(k) = \pm\sqrt{\left(2t\cos(k) + \mu\right)^2 + 4\Delta^2\sin^2(k)}. \tag{12}$$

2.2 Topological protection of edge Majorana modes

In mathematics, topology is a discipline which analyses properties of manifolds that are preserved under continuous deformations, such as genus (number of "holes") of a sphere and a torus. In physics, the concept is used for characterising topological invariants of a physical system, such as winding number (1D topological insulators), Chern number (2D), etc. [3]

Our choice of parameters in eq. (9) led us to spatially separated MZMs, but the spatial separation of MZMs does not vanish with slight changes of parameters μ, t and Δ as long as the energy band gap (of bulk spectrum) is open, that is for $|\mu| < 2t$ (eq. 12), and there is no topological phase transition.[1] [4] This connection between bulk spectrum and edge modes is known as *bulk boundary correspondence*. In other words, MZMs are topologically protected against local perturbations that do not close the energy gap. This statement could be understood physically: in fact the Hamiltonian (10) obeys particle-hole symmetry, so the energy spectrum is symmetric around zero energy. The zero energy edge mode cannot be lifted from zero energy, unless we couple it with another edge mode. To couple two MZMs we have to introduce a large enough interaction between them and by doing so destroy topologically protected states. This is only possible if we first close the band gap.

3. Braiding and quantum computation

Until now, we have discussed one topological phase at a time. MZMs can also be observed at the joints of topologically different phases, known as *domain walls*, presented in figure 2. By continuously tweaking parameters, e.g. by applying external voltage, we can move these domain walls and consequently MZMs around.

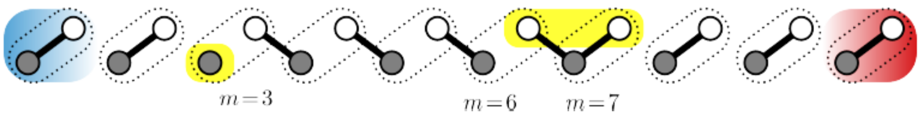


Figure 2. Fully dimerized Kitaev chain with 3 domains. Domain walls (yellow shading) host zero energy eigenstates, MZMs. These can be localised on a single site ($m = 3$), or on superposition of sites (shared between $m = 6$ and $m = 7$). Copied from [3].

If positions of two bosons are exchanged, nothing happens: $|\psi_1\psi_2\rangle = |\psi_2\psi_1\rangle$, wave function is symmetric and there is no phase shift. Contrarily, a wave function of two fermions is antisymmetric under exchange: $|\psi_1\psi_2\rangle = -|\psi_2\psi_1\rangle$. These two examples are special cases of *abelian* statistics. In the case of two quasiparticles, abelian statistics could also be $|\psi_1\psi_2\rangle = e^{i\theta}|\psi_2\psi_1\rangle$, so they could be something in between bosons and fermions.[2] But what happens if we exchange two Majorana modes? It actually turns out that Majorana modes do not obey the previous description of abelian statistics and are a part of a different class of statistics, the *non-abelian* statistics.

3.1 Non-abelian statistics

The key ingredient for non-abelian statistics is the *ground state degeneracy*. In our case that is the degeneracy of many MZMs, which can be "occupied". But can we actually speak of occupancy in a Majorana mode? We can try with analogy of occupation number for fermions $n_i = c_i^\dagger c_i$, but with Majorana operators being hermitian $\gamma_i = \gamma_i^\dagger$ and from equation (6) we deduce $\gamma_i^\dagger\gamma_i = \gamma_i\gamma_i^\dagger = 1$. Thus, in some sense, Majorana modes are always filled and empty at the same time, so that kind of counting does not give any valuable result. We return to fermionic operators, their definition of

[1] Those familiar with topology in condensed matter will recognize that in the vicinity of $|\mu| = 2t$ the winding number of band changes from 1 to 0 and the energy gap closes.

[2] These quasiparticles are called *anyons* which are only possible in two-dimensional systems.

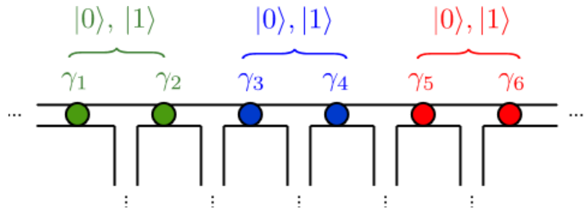
occupation number $n_i = c_i^\dagger c_i$ and corresponding basis $|s_1, s_2, ...s_N\rangle$, where s_i represents occupation number at the site i. It is useful to introduce the fermion parity operator

$$P_i = 1 - 2c_i^\dagger c_i = -i\gamma_{2i-1}\gamma_{2i} \tag{13}$$

for the pair of Majoranas, which together describe one fermion state (fig. 3).

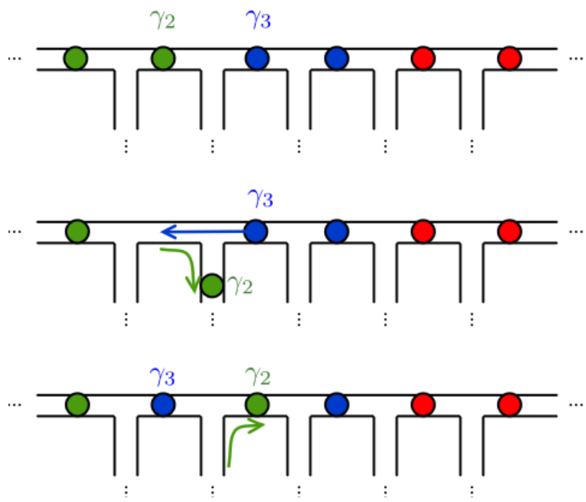
From now on we will consider a network of MZMs (γ_n) with many T−junctions in between them, as shown in figure 3. We assume that at any time MZMs are localised and far apart. MZMs are distinguishable only by their position and we will concentrate on that.

Figure 3. Two Majorana zero modes could be brought together to represent a single fermionic state with its occupancy number. Copied from [4].



Imagine now an operation of exchanging two neighbouring MZMs γ_n and γ_m by performing a time evolution represented by the trajectory shown in figure 4. The trajectory can be described by a time dependant Hamiltonian $H(t)$, $0 < t \leq T$. The time period T should be large enough, so the system does not leave the ground state and the Hamiltonian obeys the *adiabatic theorem*.

Figure 4. Exchange of two MZMs in the T−junction network. The operation is called *braiding* as timelines of two MZMs form a braid in the space-time diagram. Copied from [4].



We want to derive a unitary operator U that describes such an exchange of two MZMs. It is reasonable that U is a function of only exchanged Majorana operators γ_n and γ_m, more precisely it is a function of their product $i\gamma_n\gamma_m$, which preserves parity of fermions (eq. 13). The exponential of the imaginary unit i times hermitian operator $i\gamma_n\gamma_m$ is unitary and considering final configuration of MZMs at time T we conclude that the appropriate operator is:

$$U_{nm} = \exp\left(\pm\frac{\pi}{4}\gamma_n\gamma_m\right) = \frac{1}{\sqrt{2}}\left(1 \pm \gamma_n\gamma_m\right), \tag{14}$$

where the identity $(\gamma_n\gamma_m)^2 = -1$ was used, + and − signs correspond to clockwise and anti-clockwise (in fig. 4) exchanges of MZMs. Unitary operator U_{nm} is usually called *braiding operator* because in space-time diagram timelines of Majorana modes form entangled strands – a braid. We get an important result: in the Heisenberg picture, for a clockwise exchange, this operator's action on γ_n and γ_m is

$$\begin{aligned} \gamma_n &\rightarrow -\gamma_m, \\ \gamma_m &\rightarrow +\gamma_n. \end{aligned} \tag{15}$$

The consequence of exchange (15) is quite remarkable. To get some insight we consider the simplest non-trivial case that is an example with two fermions or equivalently with four Majorana modes. The basis is $|00\rangle$, $|11\rangle$, $|01\rangle$, $|10\rangle$, where the first digit is the occupation number of the fermionic mode $c_1^\dagger = \frac{1}{2}(\gamma_1 - i\gamma_2)$ and the second digit the occupation number of $c_2^\dagger = \frac{1}{2}(\gamma_3 - i\gamma_4)$. An arbitrary wave function can be then described by a vector of amplitudes: $|\Psi\rangle = (s_{00}, s_{11}, s_{01}, s_{10})^T$ and braiding operators U_{nm} can be then represented as 4×4 matrices. After a short calculation using identities (6) and (13) we get matrix elements for U_{12}, U_{23} and U_{34}:

$$U_{12} = \exp(\frac{\pi}{4}\gamma_1\gamma_2) = \begin{pmatrix} e^{i\pi/4} & 0 & 0 & 0 \\ 0 & e^{-i\pi/4} & 0 & 0 \\ 0 & 0 & e^{i\pi/4} & 0 \\ 0 & 0 & 0 & e^{-i\pi/4} \end{pmatrix}, \tag{16}$$

$$U_{23} = \exp(\frac{\pi}{4}\gamma_2\gamma_3) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & i & 1 \end{pmatrix}, \tag{17}$$

$$U_{34} = \exp(\frac{\pi}{4}\gamma_3\gamma_4) = \begin{pmatrix} e^{i\pi/4} & 0 & 0 & 0 \\ 0 & e^{-i\pi/4} & 0 & 0 \\ 0 & 0 & e^{i\pi/4} & 0 \\ 0 & 0 & 0 & e^{-i\pi/4} \end{pmatrix}. \tag{18}$$

Now we can clearly see where the non-abelian statistics arises: if our initial state is $|11\rangle$ and we exchange MZMs γ_2 and γ_3, that is $U_{23}|11\rangle = \frac{1}{\sqrt{2}}(|11\rangle + i|00\rangle)$, we end up in a different state, which is not related to the original one by just a phase shift. Also braiding operators do not commute, for instance: $U_{23}U_{12} \neq U_{12}U_{23}$. We only considered exchanging neighbouring MZMs, but this can be generalised, for example $U_{13} = \exp(\frac{\pi}{4}\gamma_1\gamma_3) = U_{12}^\dagger U_{23}^\dagger U_{12} = U_{12}U_{23}U_{12}^\dagger$, so we can exchange any two MZMs in the network.

3.2 Quantum computation

Equipped with the acquired knowledge we can tackle *quantum computation*. The basic idea is how to construct unitary gates to perform computations and how to store quantum data in *qubits* in a way that is robust against decoherence. A network of Majorana fermions is believed to be a possible quantum system for future *topological quantum computers* (TQC). First, Majorana fermions are topologically protected, i.e. stored information cannot be destroyed by a local perturbation. And second, we already found non-abelian unitary braiding operators that manipulate a quantum state in a nontrivial way. These operations are also topologically protected because, during braiding, MZMs are far apart at any time. However TQC based on Majorana fermions could not provide all of the unitary operations. For instance, a universal quantum computer should be able to perform the two qubit gate CNOT:

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{19}$$

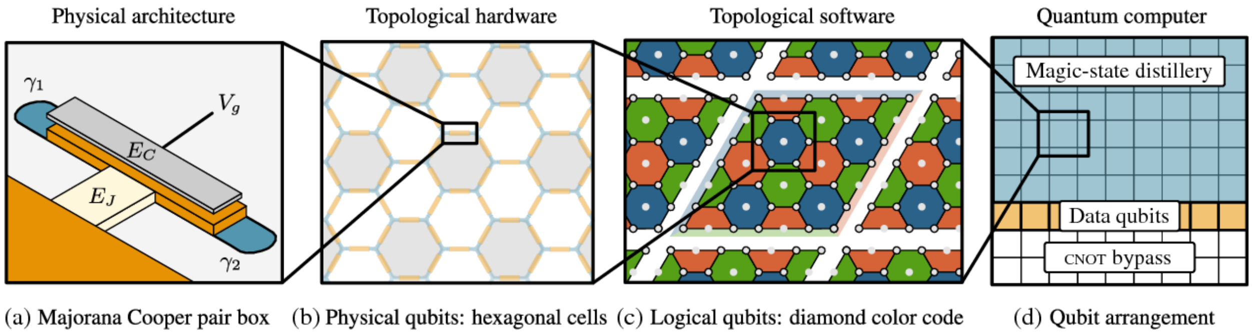


Figure 5. Basic concept of MZMs could be implemented in a complex architecture of a future topological quantum computer. Copied from [8].

where we stick to the same representation depicted in section 3.1 The reason that CNOT can not be realised just with braiding operations is that CNOT does not preserve fermion parity. [6] Universal computation could still be realised with some non-topological operations and software redundancy codes. In this context, huge progress was done in so called *color codes* whose architecture is beautiful on its own (fig. 5), but the details are well beyond the scope of this article. [7, 8]

Now we can theoretically do manipulations in quantum state, but how can we get an output? Quantum information is stored non-locally in MZMs. A Majorana mode can not be measured individually, but only in pairs as MZMs are "halves" of a fermion. Measurement is done by bringing two Majoranas together and fusing them into a fermion by adjusting the external potential in the trivial phase. The overlap of Majoranas costs additional energy $\frac{i}{2}t\gamma_{n,2}\gamma_{n+1,1} = t\left(\tilde{n} - \frac{1}{2}\right)$, expressed by the occupation number $\tilde{n}$ of the created fermion, which can be then measured.

4. Detection of Majorana fermions

In this section the detection of Majorana edge modes in a one-dimensional indium antimonide (InSb) nanowire is presented. Until now we theoretically discussed MZMs in scope of the Kitaev model and mathematically split the fermion operator into its real and imaginary part – the two Majorana operators. The other question is if the theory can be observed in nature.

4.1 From the Kitaev model to an experiment

The Kitaev model served us well to get some intuition about MZMs and their applications. Recall *Bogoliubov-de Gennes Hamiltonian* (11) for the Kitaev chain model:

$$H_{\text{Kitaev}} = (-2t\cos k - \mu)\,\tau_z + 2\Delta\tau_y \sin k. \tag{20}$$

For simplicity we will omit higher degree terms in the Taylor expansion of H_{Kitaev} and shift the chemical potential parameter $\mu \to \mu - 2t$ so that phase transition occurs at $\mu = 0$:

$$H = \left(k^2/2m - \mu\right)\tau_z + 2\Delta\tau_y k, \tag{21}$$

where m is an effective mass related to the parameter t.

We will make a few modifications to H_{Kitaev} to get a realistic Hamiltonian. First, the Hamiltonian (1) assumes spinless fermions. Well, one could think that $1/2-$spin fermions will have higher degeneracy and the rest of the story is same. But $2-$fold spin degeneracy would lead to two MZMs at each edge, which is a whole fermion, meaning there is no topological quantum computation for us. We solve this by adding the Zeeman coupling to an external magnetic field B to break the spin degeneracy:

$$H = \left(k^2/2m - \mu - B\sigma_z\right)\tau_z + 2\Delta\tau_y k, \tag{22}$$

where σ_z is the Pauli matrix acting in a spin space and as previous τ_i are the Pauli matrices acting in a particle-hole Nambu space. The idea is to choose such B that one spin will be in a topological phase and the other in a trivial phase.

In the Kitaev model we took a superconductor term that coupled neighbouring sites. In momentum space that translated to a term which was proportional to $\Delta \cdot k$ that is the $p-$wave pairing. In the real world, most of superconductors have $s-$wave pairing that has no momentum dependence. The desired pairing in a semiconducting wire could still be induced by a *proximity effect* of a $s-$wave superconductor. Here we skip a little technical step of changing the basis, also to remain bulk spectrum gapped we have to take into account spin-orbit coupling $H_{\text{SO}} = \alpha \sigma_y k$. Finally we end up with a Hamiltonian for a wire:

$$H_{\text{wire}} = \left(k^2/2m + \alpha\sigma_y k - \mu\right)\tau_z + B\sigma_z + \Delta\tau_x. \tag{23}$$

At $k = 0$ it has 4 levels with energies

$$E = \pm B \pm \sqrt{\mu^2 + \Delta^2} \tag{24}$$

and we expect the system to be topological when

$$B^2 > \Delta^2 + \mu^2. \tag{25}$$

4.2 Experimental evidence

Signatures of Majorana fermions were observed by V. Mourik et al. in ref. [9] in a hybrid superconductor-semiconductor nanowire device consisting of indium antimonide (InSb) contacted to one normal (gold) and one superconducting (niobium titanium nitride) electrode. The experimental configuration is shown in figure 6.

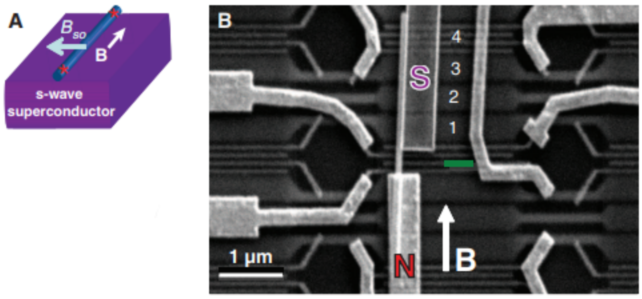


Figure 6. (A) Schematics of the InSb wire experiment. (B) Picture of the experimental layout. Adapted from [9].

Tunneling spectroscopy from a normal conductor into superconductor revealed Majorana zero mode as a *zero-bias peak* (ZBP) for a large range of applied magnetic fields B and gate voltages at a temperature of 70 mK. The main result is shown in figure 7, where distinct peaks of differential electrical conductance $\mathrm{d}I/\mathrm{d}V$ are seen at voltage $V = 0$. Magnetic field was incremented from 0 (bottom) to 490 mT (top) in 10 mT steps. The ZBP is present from $B \sim 100$ mT to ~ 400 mT, which is in agreement with the theoretically predicted topological phase transition at $B \sim 150$ mT for $\mu = 0$.

V. Mourik et al. ruled out other possible explanations, such as Kondo effect, Andreev bound states etc., and concluded that the experiment reveals the Majorana zero mode.

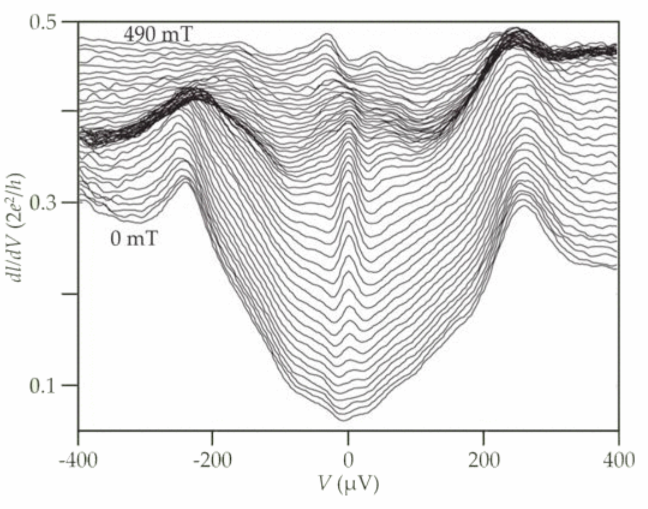


Figure 7. Zero-bias peaks in the differential conductance graph $\mathrm{d}I/\mathrm{d}V$ are seen at voltage $V = 0$. Experiment was performed at 70 mK and external magnetic fields from $B = 0$ to 490 mT in 10 mT steps. (Traces are offset for clarity, except for the lowest trace at B = 0.) The peaks at ± 250 μV correspond to the gap induced by the superconducting proximity effect. Adapted from [9].

5. Summary

The article presents the Kitaev chain model and the Majorana operators. Two fully dimerized cases are written, one topological and the other trivial. Topological protection in condensed matter is discussed.

The second section shows characteristics of the non-abelian statistics of Majorana zero modes and presents braiding operations that serve as a promising step to topological quantum computation (TQC). However braiding operations are not sufficient for a universal quantum computer.

Finally, it adds a few more steps in applying the Kitaev model to a realistic nanowire system. Experimental evidence of the Majorana zero mode in InSb nanowire is presented. Experiment shows distinct zero-bias peaks in electrical conductance in a wide range of applied external magnetic fields and confirms the existence of Majorana fermions in condensed matter physics.

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