# NON-INTEGRABILITY AND THE KAM THEOREM

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A general solution to Hamiltonian dynamical systems is in most of the cases not known. The KAM theorem qualitatively describes the orbits of a family of solvable systems after a small perturbation is applied. It helps us to understand the dynamics of complex systems and predict some characteristics of the motion.

**NE-INTEGRABILNOST IN KAM IZREK**

Splošne dinamike zahtevnih hamiltonskih sistemov v večini primerov ne poznamo. KAM izrek nam kvalitativno opisuje orbite neke družine rešljivih sistemov, če sisteme rahlo perturbiramo. Pomaga nam pri razumevanju nerešljivih sistemov in pri poznavanju nekaterih lastnosti gibanja.

## 1. Introduction

Hamilton mechanics represents a vastly used formalism in physics. In Hamiltonian mechanics, a classical physical system is described by two sets called generalized coordinates and generalized momenta. The time evolution arises as the solution of a system of first order differential equations, which are given as derivatives of the Hamiltonian function.

A lot of research has been made to fully understand general solutions of Hamilton equations. The simplest family of Hamilton systems are the integrable systems. The purpose of this article is to present one of the most widely used techniques to analyze non-integrable perturbations of integrable systems. One of the first problems which physicists tried to solve was the three body celestial problem. In 1887, Poincaré showed that there can be orbits that do not repeat themselves for certain initial conditions, so many believed that the motion is unpredictable. In the following pages I will present a theorem, which is used to predict some qualitative characteristics of such systems. For certain initial conditions it can predict that the orbits of a three body system stay closed to the unperturbed trajectories, prohibiting for example a qualitative change in Earth's orbit in a simplified model of our solar system.

A useful way to picture the solution of a Hamilton system is to present the behaviour of various trajectories, starting at different points, in the phase space. In the whole article I will adopt this point of view and I will try to analyze the changing of surfaces in the phase space that describe all solutions with the same energy.

In the first section I will briefly explain the usefulness and simplicity of integrable systems. In the second section I will show the difficulty in analyzing non-integrable systems with a perturbative approach. Finally, the third section will present the main issue of the article: the KAM theorem. I will discuss the validity and the assumptions of the method and the main results that we can get from it. The fourth section will fill the voids left by the assumptions of the KAM theorem. I will briefly discuss the behaviour of the solutions near resonances, where the conditions needed in the KAM theorem do not hold.

## 2. Integrability

In the Hamilton formalism, to each degree of freedom of a system we assign a pair of canonical variables (usually denoted as $(q_i, p_i)$). The pairs are connected by their Poisson brackets: $\{q_i, q_j\} = 0, \{p_i, p_j\} = 0, \{p_i, g_j\} = \delta_{ij}$. A Hamiltionian with $N$ degrees of freedom is said to be integrable, if exist $N$ independent integrals

$$J_i = \frac{1}{2\pi} \oint p_i \mathrm{d}q_i, \tag{1}$$

in involution, i.e., their Poisson brackets vanish for each pair ($\{J_i, J_j\} = 0$).

Equivalently, a Hamiltonian $H'_0(\vec{q}, \vec{p})$ is called integrable, if exists a transformation mapping from canonical coordinates $q_i, p_i \to \theta_i, J_i$, such that $H_0 = H_0(\vec{J})$, i.e. the Hamiltonian is dependent only on the new momenta $J_i$. These new coordinates are known as the action($J$)-angle($\theta$) variables. The equations of motion are now

$$\dot{\vec{J}} = -\frac{\partial H_0}{\partial \vec{\theta}} = 0, \tag{2a}$$

$$\dot{\vec{\theta}} = \frac{\partial H_0}{\partial \vec{J}}. \tag{2b}$$

The solutions are easy to find:

$$\vec{J} = \text{const.}, \tag{3a}$$

$$\vec{\theta} = \vec{\omega} t + \vec{c}. \tag{3b}$$

The constant action variables reduce the dimensionality of the surface of constant energy from $2N - 1$ to $N$, where $N$ is half the dimension of the phase space. The angular frequencies $\vec{\omega}$ depend only on the $\vec{J}$, so a starting point with fixed action variables will also have the angular frequencies constant.

With the help of the action-angle variables we can present an arbitrary trajectory of an integrable system as a curve on a $N$ dimensional torus. The components of the angular frequency (3b) give us the frequencies with which the trajectory winds around the $S^1$ subsets of the torus. A schematic representation of the motion on a torus is presented below:

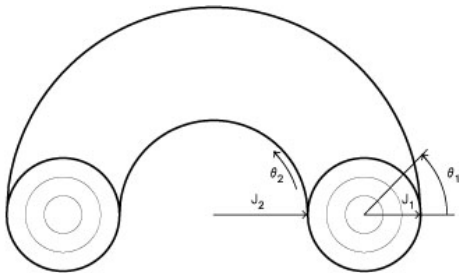

**Figure 1.** Schematic representation of an invariant torus in a 4 dimensional phase space. The radii of the torus are determined by the constants $J_1$ and $J_2$ while the the motion on the torus surface is determined by the angular frequencies $\omega_1$ and $\omega_2$ [7].

A well-known example of an integrable system is the harmonic oscillator. With a canonical change of variables we can rewrite the Hamiltonian

$$H' = \frac{mp^2}{2} + \frac{kx^2}{2} = E \tag{4}$$

in the form

$$H = (mk)^{-1/2} J, \tag{5}$$

where $J$ is the action variable $J = E(mk)^{-1/2}$. A list of some well-known integrable systems can be found in [8].

## 3. Non-integrability

From now on we will investigate the behaviour of near-integrable systems. We call a system near-integrable, if we consider the non-integrable part as a perturbation of the original integrable Hamiltonian. Such systems can be written as

$$H(\vec{J},\vec{\theta}) = H_0(\vec{J}) + \epsilon H_1(\vec{J},\vec{\theta}), \tag{6}$$

where $H_1(\vec{J},\vec{\theta})$, the non-integrable part, dependent both on the action and the angle variables.

To obtain a new Hamiltonian, dependent only on a new set of action variables, we could try to solve the Hamilton-Jacobi equation for the canonical transformation $(\vec{\theta},\vec{J}) \to (\vec{\theta}',\vec{J}')$. The generating function approach for the canonical transformation follows from the variational approach

$$\delta \int_{t_1}^{t_2} \left( \vec{J}\cdot\dot{\vec{\theta}} - H(\vec{J},\vec{\theta}) \right) = \delta \int_{t_1}^{t_2} \left( \vec{J}'\cdot\dot{\vec{\theta}}' - H'(\vec{J}',\vec{\theta}') \right). \tag{7}$$

Comparing the expression under the integral we get

$$\vec{J}\cdot\dot{\vec{\theta}} - H(\vec{J},\vec{\theta}) = \vec{J}'\cdot\dot{\vec{\theta}}' - H'(\vec{J}',\vec{\theta}') + \frac{\mathrm{d}G}{\mathrm{d}t}, \tag{8}$$

where $G$ is an arbitrary function. We can define $G$ as

$$G = -\vec{\theta}'\cdot\vec{J}' + S(\vec{\theta},\vec{J}'), \tag{9}$$

where $S$ is the 2 type generating function (see [9]), dependent only on the new momenta and the old coordinates. Since the old coordinates and new momenta are each independent, we get $\frac{\partial S}{\partial \vec{\theta}} = \vec{J}$ and so

$$H\left[\frac{\partial S}{\partial \vec{\theta}},\vec{\theta}\right] = H'(\vec{J}'), \tag{10}$$

where $S$ is the generating function, which we could expand in $\epsilon$: $S(\vec{J}',\theta) = \vec{\theta}\cdot\vec{J}' + \epsilon S_1(\vec{J}',\theta)$. In the first order we obtain

$$H_0(\vec{J}') + \epsilon\frac{\partial H_0}{\partial \vec{J}}\cdot\frac{\partial S_1(\vec{J}',\theta)}{\partial\theta} + \epsilon H_1(\vec{J}',\theta) = H'(\vec{J}'). \tag{11}$$

We can now determine $S_1$ with the condition that the left side is independent of $\theta$:

$$\vec{\omega}\cdot\frac{\partial S_1(\vec{J}',\theta)}{\partial\theta} = -H_1(\vec{J}',\theta). \tag{12}$$

Being $H_1$ and $S_1$ both periodic in the components of $\vec{\theta}$, it's useful to expand $S_1$ and $H_1$ in Fourier series

$$H_1(\vec{J}',\vec{\theta}) = \sum_{\vec{K}\neq 0} H_{1,\vec{K}}(\vec{J}')\mathrm{e}^{\mathrm{i}\vec{K}\cdot\vec{\theta}} \tag{13a}$$

$$S_1(\vec{J}',\vec{\theta}) = \sum_{\vec{K}\neq 0} S_{1,\vec{K}}(\vec{J}')\mathrm{e}^{\mathrm{i}\vec{K}\cdot\vec{\theta}}, \tag{13b}$$

where $\vec{K} = 2\pi(n_1,...,n_N)$, $n_i$ integer.

Plugging the series in (12) we get

$$S(\vec{J}', \theta) = \theta \cdot \vec{J}' + \mathrm{i}\epsilon \sum_{\vec{K} \neq 0} \frac{H_{1,\vec{K}}(\vec{J}')}{\vec{K} \cdot \vec{\omega}(\vec{J}')} \mathrm{e}^{\mathrm{i}\vec{K}\cdot\vec{\theta}}, \tag{14}$$

Immediately we can notice that the series diverges for

$$\omega_1 n_1 + \omega_2 n_2 + ... + \omega_N n_N = 0. \tag{15}$$

This can be seen by plugging (15) in the denominator of the series (14). A torus, for which we get (15), is called a resonant torus.

A good example of vanishing denominators is the motion of Jupiter and Saturn around the Sun. We can study this problem considering the attraction of the two planets as a perturbation of the motion of a single planet around the star. The ratio of the frequencies of the motion of Saturn and Jupiter around the sun is approximately $5/2$ [5], so the frequencies of the integrable system will be $\vec{\omega} \propto (5, 2)$. In the series (14) we would obtain a small denominator for $\vec{K} = 2\pi(2, -5)$, causing divergent terms in the series. The divergences mean that the two-body orbit of Jupiter-Sun will be largely disrupted if we consider the effect of Saturn.

Since rational numbers form a dense set in the real numbers, it seems that a perturbative approach such as (14) is completely useless for almost all frequencies. In 1954 A.N.Kolmogorov ([4]) proposed a theorem which significantly improved the study of perturbed dynamical systems.

In the next two chapters I will briefly explain two methods used to analyze a perturbed system. KAM theorem describes the motion far enough from the resonance (15), so that a condition explained in the next section is met. Near resonances we could get some information about the system with the help of the Poincaré-Birkhoff theorem.

## 4. KAM theorem

In a less formal manner, we can formulate the KAM theorem with terms that were previously defined. The theorem states that if, among other technicalities, the Jacobian of the frequencies is nonzero

$$\left| \frac{\partial \omega_i}{\partial J_j} \right| \neq 0, \tag{16}$$

then those tori sufficiently far from resonance which satisfy

$$|\vec{n} \cdot \vec{\omega}| \geq \gamma(\epsilon)\, |\vec{n}|^{-\tau} \qquad \lim_{\epsilon \to 0} \gamma(\epsilon) = 0, \tag{17}$$

are stable under a perturbation $\epsilon H_1$ in the limit $\epsilon \to 0$. Here $\gamma$ and $\tau$ are quantities determined by a secular perturbative approach, dependent on the number of continuous derivatives of the unperturbed Hamiltonian. The vector $\vec{n}$ has only integer components.

The proof of the theorem will not be given here; it can be found in [2].

The first condition (16) can be derived with a brief calculation.

The condition (17) in a 4 dimensional phase space implies that the frequency ratio is sufficiently far from the nearest rational number

$$\left| \frac{\omega_1}{\omega_2} - \frac{m}{s} \right| > \frac{k(\epsilon)}{s^{5/2}} \qquad \lim_{\epsilon \to 0} k(\epsilon) = 0, \tag{18}$$

for arbitrary $m, s \in \mathbb{Z}$. The derivation of the upper condition will be skipped. Note that $k(\epsilon) \neq \gamma(\epsilon)$. The ratio $5/2$ is general in the 2D case.

Naively, we can say that condition (18) is never satisfied because rational numbers are dense in the real numbers. To prove the opposite, we can estimate the total length $L$ of the intervals in $0 \leq \omega_1/\omega_2 \leq 1$, for which (18) does not hold. The length around each rational number can be estimated using equation (18). For fixed $m$ we have $s$ ration numbers $m/s < 1$. Summing over all possible $m$ we get

$$L < \sum_{s=1}^{\infty} \frac{k(\epsilon)}{s^{5/2}} s \to 0 \quad \text{for} \quad \epsilon \to 0, \tag{19}$$

where $k/s^{5/2}$ is the length of an interval around the rational number $m/s$ where (18) does not hold and $s$ in the number of integer values $m$ with $m/s \leq 1$. We see that for $\epsilon \to 0$ the length of intervals have measure zero in an interval of non-zero length. The condition is presented in the following picture

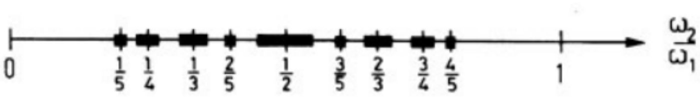


**Figure 2.** Graphic representation of the intervals around rational numbers, that contribute to $L$ [6].

A formal approach and an introduction in secular perturbation theory, which is used in the proof of the KAM theorem, can be found in [5].

A major consequence of the KAM theorem is the constraint it gives to orbits with one or two degrees of freedom. In an integrable system of lower dimension, all orbits destroyed by the perturbation are trapped inside a finite region, bordered by the neighboring non-resonant tori, as shown in picture (3). This can be easily seen considering that in a system of two degrees of freedom, an invariant tori can be seen as a 2 dimensional border of a 3 dimensional surface of constant energy. This is only possible in a 4 dimensional phase space, because the dimensionality of the torus is always equal to the number of degrees of freedom, meanwhile the energy surface is $2N - 1$ dimensional. After a small perturbation, the tori with irrational frequencies remain and confine the orbits of tori destroyed by the perturbation.

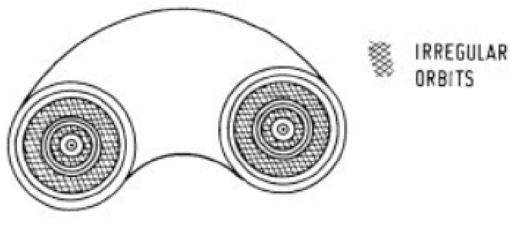


**Figure 3.** Schematic representation of regions of irregular motion, bordered by non-resonant tori [6].

In the systems with three or more degrees of freedom, $N$ dimensional tori do not represent the border of an energy surface. As a consequence, after the perturbation, the diffusion of irregular motion is possible (as seen in figure (4)). The diffusion of resonant orbits as a consequence of the lack of boundaries is called Arnold diffusion.

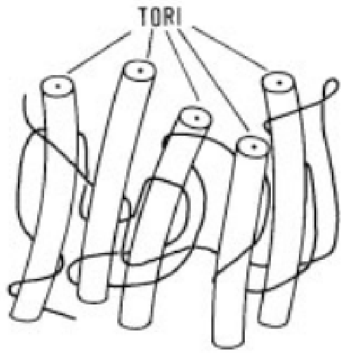


**Figure 4.** Schematic representation of Arnold diffusion. Irregular motion, presented as the trajectory that winds around non-resonant tori, is not trapped inside a finite region in the phase space [6].

## 5. The three body problem - a short example

To illustrate the usefulness of the KAM theorem in this section we will briefly analyze the three body celestial problem. We consider the motion of two planets around a massive star (so that the motion of the star can be neglected). The total Hamiltonian will have the form

$$H = \frac{|\vec{p_1}|^2}{2m_1} - \frac{K_1}{r_1} + \frac{|\vec{p_2}|^2}{2m_2} - \frac{K_2}{r_2} - \frac{K_{12}}{|\vec{r_1} - \vec{r_2}|}, \tag{20}$$

where $K_{12}$ the contribution of the last term is much smaller than the others, so that we can consider the interaction between the planets as a perturbation of the two-body system.

The first step is to find the action-angle variables of the unperturbed Hamiltonian. We follow the calculation found in [10]. We can calculate the action-angle variables independently for each planet. The problem is separable in the variables $(r, \theta, \phi)$, $H = H_r + H_\theta + H_\phi$, so we can find the 3 action variables with the help of the equation (1). Here we derive the action-angle variables for an arbitrary planes, so we omit the numbers $1, 2$ found in (20). After a short calculation we get

$$J_\phi = 2\pi \frac{\partial H_\phi}{\partial \phi} \tag{21a}$$

$$J_\theta = 2\pi \left[ \sqrt{\left(\frac{\partial H_\theta}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial H_\phi}{\partial \phi}\right)^2} - \sqrt{\left(\frac{\partial H_\phi}{\partial \phi}\right)^2} \right]. \tag{21b}$$

$$J_r = -(J_\theta + J_\phi) + \pi K \sqrt{\frac{2m}{|E|}}, \tag{21c}$$

where $K$ is the gravitational constant and $E$ the energy of the system.

The Hamiltonian is now dependent only on these action variables

$$H = -\frac{2\pi^2 m K^2}{(J_r + J_\theta + J_\phi)^2}. \tag{22}$$

The frequencies we derive from the latter equation are clearly degenerate. With a canonical transformation we can rewrite the Hamiltonian as

$$H(J) = -\frac{2\pi^2 m K^2}{J^2}, \tag{23}$$

where the frequency is $\omega = \frac{4\pi^2 mK^2}{J^3}$. More detailed derivation can be found in [10], pp. 364 − 366.

Knowing the action-angle variables of the unperturbed system, we can now check the conditions (16) and (18). The determinant (16) yields

$$\frac{12\pi^2 m_1 K_1^2}{J_1^4} \cdot \frac{12\pi^2 m_2 K_2^2}{J_2^4} \neq 0, \tag{24}$$

which is always satisfied for finite $J_1, J_2$. Next, we check the ratio of the frequencies

$$\frac{\omega_1}{\omega_2} = \frac{m_1 K_1^2 J_2^3}{m_2 K_2^2 J_1^3}. \tag{25}$$

This ratio must be sufficiently far from the nearest rational number (18). The latter equation can be also used to determine the most stable torus. The most stable torus in the $2D$ case has the ratio of frequencies equal to the most irrational number, the golden ratio. When this torus breaks, we say that the system is chaotic.

## 6. Near a resonance

The KAM theorem states that tori with frequencies sufficiently far from resonance are only slightly deformed from their initial form. In this chapter we will investigate resonant systems or systems near resonances, for which equation (17) does not hold.

Here we constrain ourself in a four dimensional phase space. It is convenient to analyze the motion on a Poincaré section, defined by the intersection of the torus with the $p_1$,$q_1$ plane (as shown below).

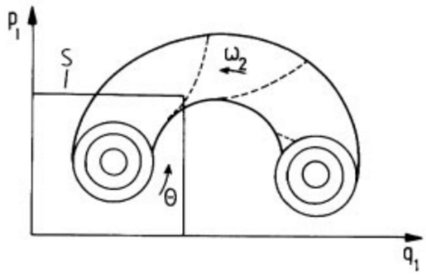


**Figure 5.** Poincaré section, defined by the intersection of the torus with the $p_1$,$q_1$ plane [6].

Before the perturbation, the motion on the section is described by

$$r_{i+1} = r_i, \tag{26a}$$

$$\theta_{i+1} = \theta_i + 2\pi\frac{\omega_1}{\omega_2}. \tag{26b}$$

where $r$ and $\theta$ represent the radius and the angle of the circle in the $q_1$,$p_1$ plane (as in the figure (5)). This map follows directly from the geometrical interpretation of the motion of integrable systems on a torus (see figure (1)). It can be shown that the frequency ratio is dependent only on the radius $r$, so that the equations (26) can be written as

$$r_{i+1} = r_i, \tag{27a}$$

$$\theta_{i+1} = \theta_i + a(r_i). \tag{27b}$$

This is Moser's twist map, which we will denote as $T(r, \theta)$. For a resonant torus we note that applying $\omega_2$ times this map leaves every point on the circle in its initial position.

If we focus on the neighboring tori of $\omega_1/\omega_2 = m/s$, we see that for tori with $s' > s$, $T^s$ will map the points on the circle clockwise, for tori with $s'' < s$, $T^s$ will map the points anti-clockwise, as shown in figure (6). Under a perturbed map $T^s_\epsilon$ these twists are preserved if $\epsilon$ is small enough. This guarantees the existence of a torus composed of fixed points of $T^s_\epsilon$.

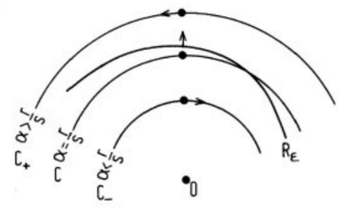


**Figure 6.** Clockwise and anti-clockwise mapping in the neighborhood of the fixed points of the mapping $T^s$. $R_\epsilon$ represents the curve formed by the points mapped after the perturbation [6].

After the perturbation, the points will be mapped also radially. The points with greater radius will be then mapped anti-clockwise and the points with smaller radius will be mapped clockwise.

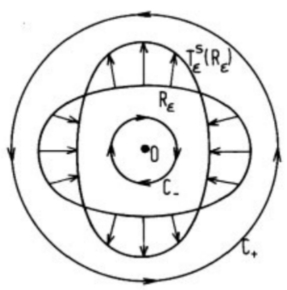


**Figure 7.** Schematic representation of the radial mapping of the perturbed map. Points mapped to a greater radius will be then mapped anti-clockwise, points mapped towards the center will be mapped clockwise [6].

After more iterations, some regions will begin to circulate around some fixed points (elliptic points), other regions instead will be repelled from the other fixed points (hyperbolic points). We get an alternating sequence of elliptic and hyperbolic points: around elliptic points new invariant tori emerge, around hyperbolic points instead chaotic motion is formed. A schematic representation of the final result is shown in figure (8):

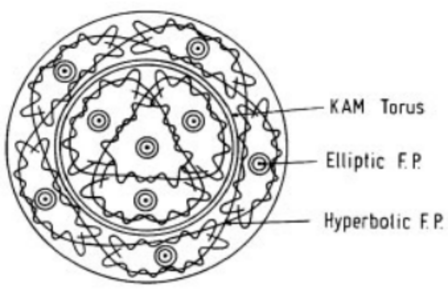


**Figure 8.** Schematic representation of regular and irregular motion in the phase space. New tori are forming around elliptic points [6].

## 7. Conclusion

In this article I presented the KAM theorem, which is of great importance in the study of perturbed dynamical systems. As mentioned in the introduction, the theorem can be applied in the study of celestial mechanics, considering for example all the planets of our solar system as a perturbation in the Earth-Sun dynamics. Given the integrable system (Earth and Sun), we can predict whether the motion of the Earth is stable or will qualitatively change its trajectory.

In the first section of the article I briefly explained some main features of the integrable systems. I explained what those systems are and briefly sketched the usefulness. In the second section I wanted to show the problems that arise when trying to solve non-integrable system with a naive perturbation approach. In the first order of $\epsilon$ we saw that the series (14) diverge for linearly dependent frequencies. In the third section was presented the main subject of the article: the KAM theorem. For the statement to hold, two conditions have to be satisfied. The first condition follows directly from the linear independence of the frequencies. I briefly sketched the meaning of the second condition (17). A detailed explanation can be found in [5]. At last I presented a method to quantitatively predict the motion after a perturbation of a resonant torus. With the introduction of a Poincaré section and Moser's twist map we can find a set of alternating elliptic and hyperbolic points, around which the motion will be regular for the former and chaotic for the latter.

The purpose of this essay was to present two ways to study non-integrable systems, focusing on the non-resonant motion. At the end I gave two ways to predict the motion after the system was perturbed: the KAM theorem, when a non-resonant torus is given, and Moser's twist map, when the orbit lies on a resonant torus.

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