

ENTANGLEMENT COOLING ALGORITHM

LUKA SKELEDŽIJA

Fakulteta za matematiko in fiziko
Univerza v Ljubljani

This paper studies entanglement dynamics in small N -qubit systems by combining theoretical analysis with numerical simulations of stochastic quantum circuits. Using the Schmidt decomposition and the Rényi entropy framework, bipartite entanglement is quantified. An entanglement “heating” procedure is implemented with random quantum gates, along with a Monte Carlo based “cooling” algorithm with Metropolis–Hastings acceptance criteria to reverse it. For systems with $2 \leq N \leq 6$, it is found that reversibility depends critically on the gate set: the Clifford generators $\{\text{CNOT}, \text{NOT}, \text{H}\}$ and $\{\text{CNOT}, \text{H}, \text{S}\}$ enable complete entropy reduction, while universal sets do not. This indicates effective irreversibility despite underlying unitary dynamics of quantum circuits.

ALGORITEM ZA HLAJENJE PREPLETENIH SISTEMOV

V prispevku je preučevana dinamika kvantne prepletenosti v majhnih sistemih z N kubiti. S pomočjo Schmidtovega razcepa in Rényijevih entropij je kvantificirana prepletenost dveh podsistemov. S pomočjo stohastičnih kvantnih vezij je implementiran postopek “gretja” (višanja entropije) prepletenega sistema. Obratni proces je realiziran z algoritmom, ki implementira Metropolis–Hastingsov kriterij, in je zaradi zmožnosti zmanjšanja entropije imenovan “hlajenje”. Za sisteme z $2 \leq N \leq 6$ je pokazano, da je reverzibilnost odvisna od nabora kvantnih vrat: Cliffordovi generatorji $\{\text{CNOT}, \text{NOT}, \text{H}\}$ in $\{\text{CNOT}, \text{H}, \text{S}\}$ omogočajo popolno zmanjšanje entropije, medtem ko univerzalne množice tega ne omogočajo. Rezultati nakazujejo pojav učinkovite ireverzibilnosti kljub unitarni dinamiki kvantnih vezij.

1. Introduction

The onset of irreversibility in physics is one of the great questions at the heart of statistical mechanics. The second principle of thermodynamics in essence states that spontaneous processes occur in one direction, namely increasing entropy or decreasing free energy. In classical mechanics, irreversibility is often attributed to chaotic dynamics, where sensitivity to initial conditions, combined with coarse-graining, leads to effective entropy production. However, in quantum mechanics, unitarity implies that slightly different initial conditions do not evolve into highly different states. Moreover, unitary evolution is always reversible, so irreversibility is strictly speaking impossible [1].

Recently, irreversible phenomena such as thermalization have been interpreted in terms of entanglement [2]. Because of entanglement, every subsystem of a pure quantum system will typically increase its entropy to give thermal expectation values to most of its observables. However, it is not entirely clear in what way entanglement is involved in the irreversible character of thermalization [1].

In this paper, we first introduce the essential concepts and then investigate irreversibility through the lens of entanglement dynamics in random quantum circuits. Starting from a product state, we apply stochastic quantum circuits that rapidly generate entanglement (“entanglement heating”), and then attempt to reverse this process using a Monte Carlo–based cooling algorithm inspired by the Metropolis–Hastings method as demonstrated in [1].

2. N -particle quantum systems

The most straightforward way to construct a state of an N -particle system is to take the tensor product of individual particle states. Such states are called separable or product states. Similarly, the basis of such a system’s Hilbert space \mathcal{H}_{AB} can be obtained by taking the tensor product of

individual particle basis states and forming the so-called product basis. The generic wave function of an N -particle system written in the product basis is

$$|\psi\rangle = \sum c_{s_1, s_2, \dots, s_N} |s_1\rangle \otimes |s_2\rangle \otimes \dots \otimes |s_N\rangle, \quad (1)$$

where s_i denotes the state of the i -th particle. The probability of measuring a given basis state is equal to the square of the absolute value of its corresponding coefficient, in accordance with Born's rule. For a more compact notation, tensor products are often omitted, and the states are written within a single ket

$$|\psi\rangle = \sum c_{s_1, s_2, \dots, s_N} |s_1\rangle |s_2\rangle \dots |s_N\rangle = \sum c_{s_1, s_2, \dots, s_N} |s_1 s_2 \dots s_N\rangle. \quad (2)$$

2.1 Entanglement in bipartite systems

Let's now consider two vectors

$$|a\rangle = \sum_i a_i |i\rangle \in \mathcal{H}_A, \quad |b\rangle = \sum_j b_j |j\rangle \in \mathcal{H}_B. \quad (3)$$

Their tensor product is given by

$$|c\rangle = |a\rangle \otimes |b\rangle = \sum_{i,j} a_i b_j |i\rangle \otimes |j\rangle = \sum_{i,j} C_{ij} |ij\rangle \in \mathcal{H}_{AB}. \quad (4)$$

We can see that the coefficients of a product state satisfy

$$C_{ij} = a_i b_j. \quad (5)$$

By rearranging the coefficients into a matrix C , we can observe that

$$C = ab^T, \quad (6)$$

which is exactly an outer product of two vectors. In general, the tensor product space \mathcal{H}_{AB} is larger than the set of simple outer products [3]. The coefficients C_{ij} can be any complex number, which means that most vectors in \mathcal{H}_{AB} cannot be factorized. If the matrix C cannot be written as ab^T , then no vectors a and b exist such that $|\psi\rangle = |a\rangle \otimes |b\rangle$. We say that a pure¹ bipartite² system is

$$\text{entangled} \iff C \neq ab^T \iff \text{rank}(C) > 1. \quad (7)$$

For mixed states or multipartite systems, entanglement cannot in general be characterized by a simple matrix-rank condition. One must instead use density matrices and criteria such as the positive partial transpose (PPT) criterion, entanglement witnesses, or other separability tests.

2.2 Schmidt decomposition

In the previous section, we have discussed entanglement between two subsystems A and B . For an N -particle system, it is possible to choose many different subsystems by assigning $r < N$ particles to subsystem A and $N - r$ particles to subsystem B . We will now show how to quantify the amount of entanglement for a specific bipartition, an essential method used in our entanglement cooling algorithm.

¹A pure state is a quantum state that cannot be expressed as a statistical mixture of other states. It is represented by a single normalized vector $|\psi\rangle$ in Hilbert space, or equivalently by a rank-one density matrix $\rho = |\psi\rangle\langle\psi|$.

²A system composed of two subsystems.

We start by writing a generic wave function in the Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ as shown before

$$|\psi\rangle = \sum_{\alpha,\beta} c_{\alpha\beta} |\alpha\rangle \otimes |\beta\rangle. \quad (8)$$

We recognize that the coefficients $c_{\alpha\beta}$ can be rewritten as a matrix C . In general, C is not a square matrix as its size is determined by the sizes of its subsystems (which are in turn determined by parameters r and N). For any real and complex matrix, we can perform something called Singular Value Decomposition (SVD), which can be thought of as a more general form of eigen-decomposition for non-square matrices. By performing SVD on a matrix M , we can decompose it into a product of a unitary, diagonal and another unitary matrix [4].

$$M = U\Sigma V^\dagger. \quad (9)$$

The values of the diagonal matrix Σ are known as singular values. If the singular values appear in descending order, then they are uniquely determined up to permutation of equal values. From our coefficient matrix we therefore obtain

$$c_{\alpha\beta} = \sum_k U_{\alpha k} \lambda_k V_{k\beta}^\dagger, \quad (10)$$

where $U_{\alpha k}$ and $V_{k\beta}^\dagger$ are elements of unitary matrices, and λ_k are the singular values. We rewrite the wave function of the composite system, and this time we use the singular value decomposition for the coefficient matrix:

$$|\psi\rangle = \sum_{\alpha,\beta,k} \lambda_k U_{\alpha k} |\alpha\rangle \otimes |\beta\rangle V_{k\beta}^\dagger. \quad (11)$$

Since the matrices U and V^\dagger are unitary, we can define new bases for subsystems A and B

$$|u_k\rangle = \sum_\alpha U_{\alpha k} |\alpha\rangle, \quad |v_k\rangle = \sum_\beta V_{k\beta}^\dagger |\beta\rangle. \quad (12)$$

The wave function of the system can now be written in the new subsystem bases

$$|\psi\rangle = \sum_k \lambda_k |u_k\rangle |v_k\rangle, \quad (13)$$

This process is also known as the Schmidt decomposition. The theorem behind it guarantees us that if $|\psi\rangle$ is a pure state of a composite system AB , then there exist orthonormal states $|u_k\rangle$ for system A , and orthonormal states $|v_k\rangle$ of system B such that equation (13) holds. Coefficients λ_i are non-negative real numbers satisfying $\sum_i \lambda_i^2 = 1$ known as Schmidt coefficients. This allows us to rewrite $|\psi\rangle$ in a form that uses only one index k , that runs over all non-zero coefficients λ_i . Consequently, the number of orthogonal states $|v_k\rangle$ and $|u_k\rangle$ is always equal to k (even for completely asymmetric split sizes), which is a number known as the Schmidt rank. For pure bipartite states, the rank of coefficient matrix C is the same as Schmidt rank [5, 6].

2.3 Rényi entropies for bipartite systems

We will now use Schmidt decomposition to quantify the amount of entanglement. By defining the reduced density matrix of subsystem A as

$$\rho_A = \text{Tr}_B(|\psi\rangle\langle\psi|) \quad (14)$$

and inserting the Schmidt form $|\psi\rangle = \sum_k \lambda_k |u_k\rangle |v_k\rangle$, we obtain

$$\rho_A = \sum_k \lambda_k^2 |u_k\rangle \langle u_k|. \quad (15)$$

We can see that eigenvalues of ρ_A are simply $p_k = \lambda_k^2$, and the number of nonzero p_k equals the Schmidt rank $r_S = k$, because k has not changed [6]. As we have mentioned before, separable states have $r_S = 1$, while entangled states satisfy $r_S > 1$. A good choice for quantification would therefore be a monotone function where $f(r_S)|_{r_S=1} = 0$, which leads us to the logarithmic function

$$S_0(\rho_A) := \log \text{rank}(\rho_A) = \log r_S. \quad (16)$$

We call this function S_0 . It distinguishes product from entangled states and counts how many Schmidt modes are populated, but it ignores how the weights p_k are distributed. To quantify how much entanglement is present, we need functions that depend on the full spectrum $\{p_k\}$, otherwise our function is extremely sensitive to noise. This leads us to the Rényi family of entropy functions [7], with S_0 being the first member. The α -th Rényi entropy is defined as

$$S_\alpha(\rho_A) = \frac{1}{1-\alpha} \log \left(\sum_i p_i^\alpha \right). \quad (17)$$

The von Neumann entropy arises as the continuous limit of the Rényi family for $\alpha \rightarrow 1$. Using L'Hôpital's rule, we can derive the von Neumann entropy as

$$S_1(\rho_A) := -\text{Tr}(\rho_A \log \rho_A). \quad (18)$$

For pure bipartite states, the von Neumann entropy of the reduced density matrix is the standard measure of entanglement. It vanishes if and only if the state is separable, and it increases as the Schmidt coefficients become more uniformly distributed. In particular, it attains its maximal value $\log r_S$ when subsystem A is maximally entangled with B . For pure bipartite states, the reduced density matrices ρ_A and ρ_B share the same nonzero eigenvalues $\{p_i\}$. Consequently, it can be shown [5]

$$S_1(\rho_A) = S_1(\rho_B). \quad (19)$$

The entanglement entropy is therefore symmetric with respect to the bipartition.

Although the global state $|\psi\rangle$ contains complete information, an observer with access only to subsystem A describes it by the mixed state ρ_A . The entropy $S_1(\rho_A)$ therefore measures the information that is inaccessible locally due to entanglement with subsystem B . For two particles with opposite spins, Bob would know his particle's spin if Alice's measurement result were available to him; however, because he only has local access to his subsystem, that information is encoded in non-local correlations and is inaccessible to him alone.

3. Quantum gates and circuits

In quantum mechanics, the evolution of a closed system is described by a unitary operator

$$U^\dagger U = U U^\dagger = \mathbb{I}. \quad (20)$$

Quantum gates are unitary operators acting on one or more qubits. They represent the elementary building blocks of quantum circuits, analogous to logical gates in classical computation. A gate set \mathcal{I} is called universal if any unitary operator on N qubits can be approximated to arbitrary precision by a finite sequence of gates drawn from \mathcal{I} . Two structural requirements are necessary

for universality: (i) the ability to approximate arbitrary single-qubit rotations, generating a dense subset of $SU(2)$, and (ii) the presence of at least one entangling two-qubit gate. Without entangling gates, separability is preserved and multi-qubit entanglement cannot be generated. Conversely, it can be shown that any set containing all single-qubit unitaries together with one entangling two-qubit gate is universal [5].

3.1 Common Quantum Gates

Some of the most common quantum gates are displayed in Table 1. A single-qubit gate acting on

Name	Action on Basis States	Matrix Representation
Pauli- X (NOT)	$ 0\rangle \leftrightarrow 1\rangle$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Pauli- Y	$ 0\rangle \mapsto i 1\rangle$ $ 1\rangle \mapsto -i 0\rangle$	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$
Pauli- Z	$ 0\rangle \mapsto 0\rangle$ $ 1\rangle \mapsto - 1\rangle$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Hadamard (H)	$H 0\rangle = \frac{1}{\sqrt{2}}(0\rangle + 1\rangle)$ $H 1\rangle = \frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$	$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
Phase ($S = P_{\pi/2}$)	$ 0\rangle \mapsto 0\rangle$ $ 1\rangle \mapsto i 1\rangle$	$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$
T Gate ($P_{\pi/4}$)	$ 0\rangle \mapsto 0\rangle$ $ 1\rangle \mapsto e^{i\pi/4} 1\rangle$	$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$
Controlled-NOT (CNOT)	$ c, t\rangle \mapsto c, t \oplus c\rangle$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Table 1. Common quantum gates used in quantum circuits [5].

the i -th qubit has the form

$$U_i = \mathbb{I} \otimes \dots \otimes U \otimes \dots \otimes \mathbb{I}. \tag{21}$$

If the initial state is separable ($|\psi\rangle = |a\rangle \otimes |b\rangle$), then applying local unitary operators yields

$$(U_A \otimes U_B)|\psi\rangle = (U_A|a\rangle) \otimes (U_B|b\rangle), \tag{22}$$

which remains separable. Therefore, local (single-qubit) gates cannot generate entanglement. Entanglement can only be generated by gates that act nontrivially on more than one qubit simultaneously. A fundamental example is the controlled-NOT (CNOT) gate. Consider the product state

$$|\psi_0\rangle = |+\rangle \otimes |0\rangle, \tag{23}$$

where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Applying a CNOT gate yields

$$\text{CNOT}|\psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \tag{24}$$

which is a maximally entangled Bell state [3].

3.2 Stochastic circuits and entanglement heating

A quantum circuit consists of a sequence of quantum gates applied in discrete time steps,

$$|\psi(t)\rangle = U_t U_{t-1} \cdots U_1 |\psi(0)\rangle. \quad (25)$$

In the following sections, we consider stochastic quantum circuits, where at each step a randomly chosen gate acts on randomly selected qubits. Starting from a product state, repeated application of quantum gates will rapidly generate entanglement across the system. This process is referred to as *entanglement heating*; the systematic growth of bipartite entanglement under random unitary evolution. Because quantum evolution is unitary, it is fundamentally reversible. Any quantum gate U admits an inverse $U^{-1} = U^\dagger$ [1].

4. Entanglement cooling algorithm

Although unitary evolution is strictly reversible at the microscopic level, recovering the initial product state requires applying the exact inverse circuit. If the sequence of applied gates is unknown, the inverse operation cannot be constructed directly. Our cooling algorithm therefore does not implement true time reversal. Instead, it performs a stochastic search in circuit space, using entanglement entropy as a cost function to guide the system toward lower-entanglement states. The observed irreversibility does not originate from fundamental non-unitarity, but from the computational difficulty of reconstructing the exact inverse transformation.

Pseudo-code implementation of the entanglement cooling algorithm is shown in Algorithm 1. The cooling algorithm works by randomly selecting gates from the universal set and applying them to the entangled state. Unlike heating, where all gates are accepted, cooling uses a Metropolis–Hastings-like acceptance criterion to navigate the exponentially large space of possible gate sequences. Gates are either accepted or rejected based on the entropy change rule. The parameter β (in units of entropy) acts as control (or balance) between exploration and exploitation (reducing the entropy). The algorithm continues until the entanglement entropy reaches zero or a maximum number of iterations is exceeded [1].

Algorithm 1 Entanglement Cooling Algorithm [1]

```

1:  $S_{\text{old}} \leftarrow$  entanglement entropy of  $\psi$ 
2: while  $S_{\text{old}} > 0$  do
3:   Apply gate at random:  $\psi \leftarrow \psi_{\text{new}}$ 
4:    $S_{\text{new}} \leftarrow$  entanglement entropy of  $\psi$ 
5:   if  $S_{\text{new}} > S_{\text{old}}$  then
6:      $r \leftarrow x \in [0, 1]$ 
7:     if  $r > \exp[\beta(S_{\text{new}} - S_{\text{old}})]$  then
8:       Undo gate:  $\psi \leftarrow \psi_{\text{old}}$ 
9:     end if
10:  end if
11:   $S_{\text{old}} \leftarrow S_{\text{new}}$ 
12: end while

```

4.1 Entropy calculations

The cooling algorithm is computationally demanding on classical hardware, as each Monte Carlo step requires exact diagonalization and the Hilbert space dimension grows exponentially with system

size. For each state, the entanglement entropy is evaluated across the $N - 1$ possible bipartitions. These bipartitions are constructed by arranging the qubits on a line and considering all possible cuts between them.

5. Simulations

We created and ran the simulations ourselves for small system sizes from $N = 2$ up to $N = 6$. They were implemented in PYTHON, using NUMPY for linear algebra and SCIPY for eigenvalue decomposition of reduced density matrices. The quantum state is represented as a dense complex vector of dimension 2^N , and gate operations are constructed via Kronecker products of single-qubit matrices. Since each step of the algorithm requires computing the entanglement entropy — which involves diagonalizing reduced density matrices of dimension up to $2^{\lfloor N/2 \rfloor}$ for each of the $N - 1$ bipartitions — the computational cost per step scales exponentially with N . This imposes a trade-off between statistical quality (number of independent realizations) and system size: for $N = 6$, a single cooling trial completes in seconds (on a Macbook Pro with 18 GB of RAM and M3 Pro SOC), whereas for $N = 8$ it requires on the order of minutes.

All simulations share the following protocol. The initial state is always the product state $|0\rangle^{\otimes N}$. In the heating phase, 150 random gates are applied to generate an entangled state. In the cooling phase, the algorithm runs for up to 1000 steps with a success threshold of $S < 0.01$. For each parameter configuration (system size or inverse temperature), we perform 25 independent realizations with different random seeds. Error bars in all figures represent one standard deviation across these realizations. Increasing the number of realizations beyond 25 would improve statistical precision, but is limited by computational cost: a single realization for $N = 6$ already requires several seconds, so 25 trials per configuration represents a practical compromise. Computational results are presented below.

In the first phase of the simulations we compare different suggested gate sets. As can be seen in Fig. 1a, applying random gates from all tested gate sets increases the entropy, although with slightly different dynamics. However, as shown in Fig. 1b, only the two Clifford gate sets, $\mathcal{I}_1 = \{\text{CNOT}, \text{NOT}, \text{H}\}$ and $\mathcal{I}_2 = \{\text{CNOT}, \text{H}, \text{S}\}$, achieve a complete reduction in entropy, while the sets containing the T gate do not. We discuss this in Section 6.

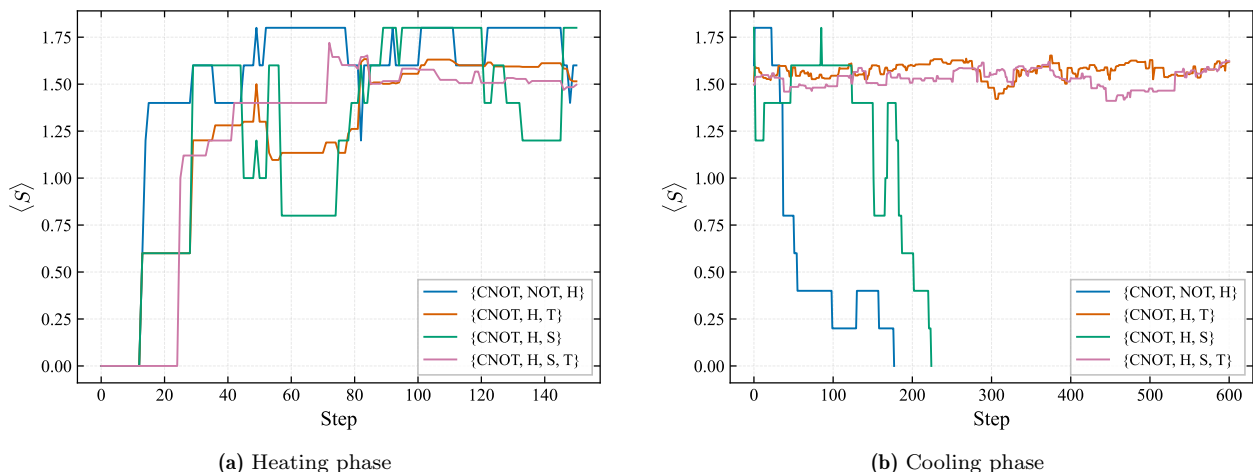


Figure 1. Entanglement heating (150 steps) and cooling (600 steps, $\beta = 5$) for $N = 6$ qubits (single realization, initial state $|0\rangle^{\otimes 6}$). Four gate sets are compared; only the two Clifford sets ($\{\text{CNOT}, \text{NOT}, \text{H}\}$ and $\{\text{CNOT}, \text{H}, \text{S}\}$) achieve complete entropy reduction, while the sets containing the T gate do not.

In the next part, we compare the average cooling time and the number of steps required for disentanglement for system sizes ranging from $N = 2$ to $N = 6$. As shown in Fig. 2a and Fig. 2b, the

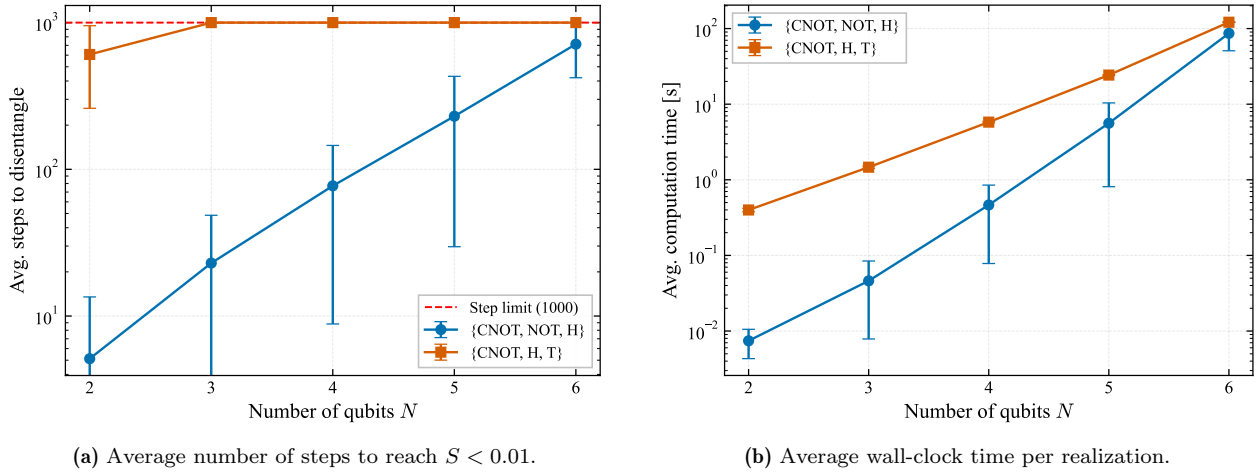


Figure 2. Disentangling speed vs. system size for both gate sets ($\beta = 5.0$, 25 independent realizations per data point, max 1000 steps, initial state $|0\rangle^{\otimes N}$). Error bars show $\pm 1\sigma$. The dependence on N is approximately exponential (note the logarithmic y -axis).

dependence is approximately exponential (note the logarithmic y -axis). This behavior is expected, since the cooling procedure relies on exact diagonalization and the Hilbert space dimension grows as $\dim \mathcal{H} = 2^N$, leading to exponential scaling with system size.

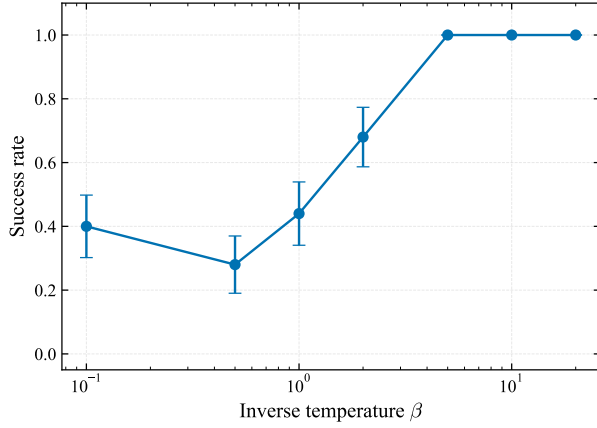
Finally, we assess the role of the parameter $\beta = \frac{1}{k_B T}$, which controls the balance between exploration and exploitation in our Metropolis–Hastings–like algorithm. In the limit $\beta \rightarrow \infty$ (corresponding to $T \rightarrow 0$), the algorithm becomes greedy: only entropy-reducing moves are accepted, since the acceptance probability $\exp[\beta(S_{\text{new}} - S_{\text{old}})]$ suppresses any increase in entropy. In contrast, when $\beta \rightarrow 0$, the acceptance probability approaches unity and the algorithm effectively performs a random walk in state space, without entropy reduction. The optimal performance is therefore expected at intermediate values of β , where the algorithm allows limited exploration while performing exploitation. This behavior is shown in Fig. 3. A trial is considered successful if the algorithm reduces the entanglement entropy below $S < 0.01$ within 1000 steps. For each value of β , we perform 25 independent realizations starting from independently heated states. We observe that the success rate (Fig. 3a) increases with β , while both the average number of steps (Fig. 3b) and the required runtime (Fig. 3c) decrease with β becoming larger.

6. Discussion

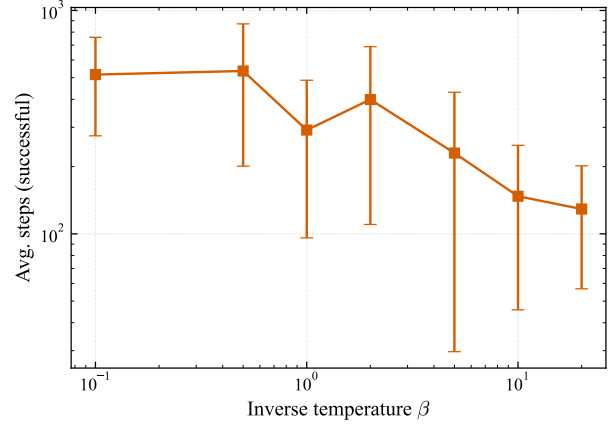
The simulation results show that the two Clifford gate sets succeed at cooling, while the two sets containing the T gate do not. We will try to provide some insights into why this could be the case.

All three gates in the set $\mathcal{I}_1 = \{\text{CNOT, NOT, H}\}$ are self-inverse: $X^2 = H^2 = \text{CNOT}^2 = \mathbb{I}$. This means that any gate applied during heating can be undone by applying the same gate to the same qubits, giving the cooling algorithm a higher probability of reversing a heating step with a single random guess. For the set $\mathcal{I}_2 = \{\text{CNOT, H, S}\}$, the situation is slightly less favorable. Gate S is not self-inverse ($S^4 = \mathbb{I}$, so $S^{-1} = S^3$), but the inverse is still a short sequence within the Clifford group. In contrast, the T gate satisfies $T^8 = \mathbb{I}$, meaning that $T^{-1} = T^7$. While the cooling algorithm does not need to retrace the exact inverse circuit, the self-inverse property nonetheless provides a significant probabilistic advantage since each random gate application is more likely to reduce entanglement when every generator is its own inverse. More formally, both \mathcal{I}_1 and \mathcal{I}_2 generate subgroups of the finite Clifford group, so the cooling algorithm performs a random walk on a finite state space.

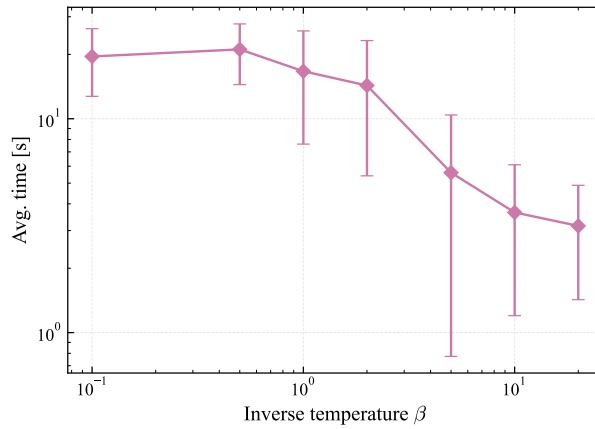
Entanglement cooling algorithm



(a) Success rate (fraction of 25 trials reaching $S < 0.01$). Error bars show the binomial standard error $\sqrt{p(1-p)/25}$.



(b) Average number of steps for successful trials ($\pm 1\sigma$).



(c) Average wall-clock time per realization ($\pm 1\sigma$).

Figure 3. Dependence of disentangling performance on the inverse temperature β for the reversible gate set $\mathcal{I} = \{\text{CNOT}, \text{NOT}, \text{H}\}$ ($N = 5$, 25 independent realizations per β value, max 1000 steps, initial state $|0\rangle^{\otimes 5}$).

Adding the T gate promotes the gate set to universality (Clifford + T), meaning that arbitrary unitaries in $U(2^N)$ can be approximated to any desired precision. The reachable state space thus becomes dense in the full Hilbert space, and in this exponentially large space, spontaneous return to the initial state becomes overwhelmingly unlikely. This distinction between gate sets is also reflected in the level spacing statistics of the corresponding circuit ensembles. Clifford circuits exhibit Poisson statistics (characteristic of integrable systems), while universal circuits display Wigner-Dyson statistics (characteristic of quantum chaos), as analyzed in [1].

7. Conclusion

In this work, we combined the theoretical framework of bipartite entanglement with numerical simulations of stochastic quantum circuits to study the emergence of effective irreversibility in small N -qubit systems. After introducing tensor products, the Schmidt decomposition, and Rényi entropies, we established how entanglement in pure states can be quantified through the eigenvalues of the reduced density matrix. In particular, the von Neumann entropy provides a natural measure of bipartite entanglement.

Starting from separable product states, random quantum circuits generate rapid entanglement growth (“heating”), reflected in increasing subsystem entropy. Although the global evolution is unitary and therefore reversible in principle, recovering the initial state requires reconstructing the

exact inverse circuit. Instead, we implemented a Monte Carlo–based cooling algorithm guided by entanglement entropy.

We find that successful disentanglement depends strongly on the chosen gate set. The sets $\mathcal{I}_1 = \{\text{CNOT}, \text{NOT}, \text{H}\}$ and $\mathcal{I}_2 = \{\text{CNOT}, \text{H}, \text{S}\}$ allow complete entropy reduction, while other sets do not. The cooling time scales approximately exponentially with system size, consistent with the exponential growth of Hilbert space dimension $\dim \mathcal{H} = 2^N$. The Metropolis–Hastings parameter β controls the balance between exploration and entropy reduction, with larger values improving convergence.

Overall, our results illustrate how entanglement dynamics can give rise to effective irreversibility, even though the underlying quantum evolution remains strictly unitary.

REFERENCES

- [1] D. Shaffer, C. Chamon, A. Hamma, and E. R. Mucciolo, *Irreversibility and entanglement spectrum statistics in quantum circuits*, J. Stat. Mech. **2014** (2014), P12007.
- [2] S. Popescu, A. J. Short, and A. Winter, *Entanglement and the foundations of statistical mechanics*, Nature Phys. **2** (2006), 754–758.
- [3] A. Pathak, *Elements of Quantum Computation and Quantum Communication*, Taylor & Francis, 2013.
- [4] T. Šket, *Kvantna prepletenost kot mera informacije*, Matrika **12** (2025), no. 2, <https://matrika.fmf.uni-lj.si/letnik-12/stevilka-2/sket.html>.
- [5] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, 10th Anniversary ed., Cambridge Univ. Press, 2012.
- [6] M. Kumar, *On properties of Schmidt decomposition*, preprint, arXiv:2411.05703 [quant-ph] (2024).
- [7] M. Ozawa and N. Javerzat, *Perspective on physical interpretations of Rényi entropy in statistical mechanics*, preprint, arXiv:2404.06436 [cond-mat.stat-mech] (2024).