PERCOLATION THEORY

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This article introduces percolation theory, which is a mathematical model that is useful in the description of many physical phenomena. It is also a prime example of a model with a sharp phase transition. Firstly, it explains the theoretical background and defines the relevant parameters, quantities, and phenomena, including the spontaneous appearance of self-similarity and fractals at the percolation transition. The latter is explained using the techniques of renormalization. Finally, it presents some applications of percolation theory, and in detail illustrates its use in describing the spontaneous magnetization of a random, dilute spin $\frac{1}{2}$ Ising model of magnetism in various dimensions.

PERKOLACIJSKA TEORIJA

Članek predstavi perkolacijsko teorijo, ki je matematični model, uporaben za opis številnih fizikalnih pojavov. Je tudi tipičen primer modela z ostrim faznim prehodom. Najprej pojasni teoretično ozadje in definira ustrezne parametre, količine in pojave, vključno s spontanim pojavom samopodobnosti in fraktalov pri perkolacijskem prehodu. Slednje je razloženo s tehnikami renormalizacije. Na koncu predstavi nekaj primerov uporabe perkolacijske teorije ter podrobno ponazori njeno uporabo pri opisu spontane magnetizacije naključnega, razredčenega Isingovega modela magnetizma s spinom $\frac{1}{2}$ v različnih dimenzijah.

1. Introduction

Percolation theory is a class of simple mathematical models of random disordered systems that display phase transitions. These are not exactly solved models, but they often give good insight when applied to real physical systems. Observables in such systems are dependent mainly on the statistics of the size and structure of clusters that are spontaneously formed, which is common in many physical systems. At the phase transition, these clusters grow increasingly large and ultimately become infinite (in the case of an infinite lattice). At the phase transiion observables diverge with universal critical exponents, just as in second-order thermodynamic phase transitions, and the structure of clusters becomes fractal. We begin by describing the percolation models and then introduce the necessary terminology, quantities, and techniques used to describe them. We will justify the appearance of fractals at the percolation transition using renormalization group techniques.

We will give a short description of a few disordered physical systems where percolation theory can be used. For example, modelling the spread of infections or information on social networks [1, 2]. We can use it to calculate fluid flow in porous media or electric current in materials with clusters of conducting particles, for example [3]. In special cases, it can also describe the phase transitions of ordered (non-random) materials, e.g., magnetic systems, when these are dual to a random percolation model [4]. We will illustrate the use of this theory on a practical problem in magnetism using numerical simulations.

We can also introduce a correlation between the occupancies of different sites, where, for example, the likelihood of a site being occupied next to another occupied site is either higher or lower than if the other site was unoccupied. We can also study bond percolation, where sites are always present, but connections (bonds) between them are either occupied or not. These generalizations make percolation theory a powerful tool for modeling a wide variety of physical and other complex

systems, which is why it remains a highly active field of research. This is exemplified by the awarding of the Fields Medal in 2022 for work that revealed connections between percolation theory and the theory of random walks.

2. Percolation theory

2.1 Definitions and terminology

We define the parameter p as the probability that a lattice site is occupied. It is usually called the *concentration* or *occupation probability*. Next, we define the term *cluster* as a connected group of occupied lattice sites on the lattice, surrounded by unoccupied sites. In other words, we can get from any one site in a cluster to any other by hopping between nearest-neighbouring occupied sites, but we cannot get to a different cluster in this way. We denote the average number of clusters of size s per lattice site as $n_s(p)$. Here, the definition is taken per lattice site so that $n_s(p)$ is independent of the size of the lattice, at least in the limit of a large (ideally, infinite) lattice. We will use the term *s-cluster* to mean a cluster of size s.

Next, we define p_c, the *percolation threshold*, as the lowest concentration p at which an infinite-sized cluster appears in the lattice with probability one. When such a cluster is present, we say that the lattice percolates. Here, p_c is exactly defined only in the limit of an infinite lattice and we can (at least numerically) determine its value for most common lattices in the literature. Some known values of p_c are collected in Table (1), taken from Ref. [5]. When we are working with a finite lattice we say that it percolates when a cluster appears that spans the lattice from a given edge to an edge opposite of it.

For simplicity, we will first look at the percolation (random occupation of sites) in a 1D (one-dimensional) infinite lattice (i.e., a chain of sites), as shown in Figure (1). We can observe that in this case $p_c = 1$ since an infinite cluster cannot form if even a single site is empty. When we go to lattices in higher dimensions it turns out that the value of p_c is lower than 1, which can be understood by there being many more possible configurations of occupied sites that give rise to an infinite cluster.

Figure 1. Infinite 1D lattice. The probability for a site to be occupied is p, in which case it is depicted as a full circle, while the probability for a site to be unoccupied is $1-p$, in which case it is depicted as an empty circle. In the example shown, we can see four 1-clusters, one 3-cluster and one 4-cluster. Inspired by Ref. [5].

lattice	# nn	p_c
1D	2	1
2D Honeycomb	3	0.6962
2D Square	2	1
2D Triangular	6	1/2
3D Diamond	4	0.43
3D Simple Cubic	6	0.3116
3D Body Centered Cubic	8	0.246
3D Face Centered Cubic	12	0.198

Table 1. The percolation thresholds p_c for typical lattices. # nn denotes the number of nearest neighbours a site has in the given lattice. 2D stands for two-dimensional, 3D for three-dimensional. Values are taken from Ref. [5].

Throughout the article, we will (unless specified otherwise) be working with a discrete lattice

(e.g., a 1D chain of sites or a 2D (two-dimensional) square lattice) in which sites are randomly occupied or unoccupied independently of neighbouring sites. This is called uncorrelated site percolation.

2.2 Critical exponents

Critical exponents are important quantities that help us understand the behaviour of our system close to its phase transition. Using the 1D chain example from Figure 1, where $p_c = 1$, we introduce the *characteristic cluster size* s_ξ and its critical exponent σ. We calculate the probability for a cluster of size exactly s (i.e., s occupied sites surrounded by 2 unoccupied sites) to appear in the limit $L \to \infty$, where L is the number of sites on the chain,

$$n_s(p) = (1-p)\, p^s\, (1-p) = (p_c - p)^2 \exp\left(-\frac{s}{s_\xi(p)}\right), \tag{1}$$

where $p_c = 1$ and $s_\xi(p) = -1/\ln(p)$ is the characteristic cluster size (in the sense of the characteristic number of sites in a cluster). Taylor expanding s_ξ around the percolation threshold $p \approx p_c$ we get

$$s_\xi(p) \approx \frac{1}{p_c - p}.$$

From this expression, we can see that the characteristic cluster size diverges as a power law around the critical concentration for percolation p_c. It can be shown that this also happens in higher dimensions where $p_c < 1$, the only difference being in the (critical) exponent of the power-law divergence [5]. Namely, the critical exponent σ for the characteristic cluster size is defined via

$$s_\xi(p) \propto |p_c - p|^{-1/\sigma} \qquad \text{for } p \to p_c.$$

As we saw, $\sigma = 1$ in 1D. It turns out that this exponent depends only on the dimensionality of the lattice (Table 2) [5]. This is called *universality* and we say that the percolation transitions of all lattices of the same dimensionality fall into the same *universality class* with its characteristic critical exponents.

Let us also introduce another critical exponent, β. Firstly, we notice that since $n_s(p)$ is the concentration of clusters of size s per lattice site, and each such cluster encompasses s lattice sites, the probability that a lattice site picked at random is part of a finite ($s < \infty$) s-cluster is $sn_s(p)$, while the probability that it is unoccupied (and thus not part of any cluster) is $1 - p$. Thus, the conditional probability, that an occupied site picked at random is part of a cluster of size s is given by

$$w_s(p) = \frac{sn_s(p)}{p}.$$

Here, $\sum_{s=1}^{\infty} w_s(p) = 1$ below the percolation threshold, $p < p_c$, while above it we have a finite probability $P(p)$ [which one could also denote $w_\infty(p)$] which we call the *strength of the infinite cluster*, that the randomly-picked occupied site is not part of any finite cluster, but rather is part of an infinitely large cluster. In other words, $\sum_{s=1}^{\infty} w_s(p) + P(p) = 1$. One can show [5], that in general

$$P(p) \propto (p - p_c)^\beta \qquad \text{for } p \to p_c^+, \tag{2}$$

where we approach p_c from above, while $P(p) = 0$ for $p < p_c$ (see Figure 2). The strength $P(p)$ is thus an *order parameter* of the percolation transition and is the main quantity in percolation theory. The critical exponent β also only depends on the dimensionality of the lattice (see Table 2).

Next, we define the *mean cluster size* $S(p)$, and its associated critical exponent γ. Firstly,

$$S(p) = \sum_{s=1}^{\infty} s w_s = \sum_{s=1}^{\infty} \frac{s^2 n_s(p)}{p}.$$

For example, for a 1D chain we can use Eq. (1) to calculate

$$S(p) = \frac{1+p}{1-p} = \frac{p_c + p}{p_c - p}.$$

For $p \approx p_c = 1$ we can expand this as

$$S(p) \approx \frac{2p_c}{p_c - p} \propto (p_c - p)^{-1},$$

which again diverges as a power law. In general, we define the critical exponent γ

$$S(p) \propto |p_c - p|^{-\gamma} \qquad \text{for } p \to p_c.$$

In 1D, $\gamma = 1$, as we have shown, while it is different for higher dimensions (see Table 2).

Finally, we define the *correlation function* or *pair connectivity* $g(\mathbf{r}, p)$ of finite clusters as the probability that a site displaced by $\mathbf{r}$ from an initial, randomly picked, occupied site is also occupied and is part of the same finite cluster as the initial site. For $p \neq p_c$, the correlation function decays exponentially at large distances with a characteristic distance $\xi(p)$, which we call the *correlation distance*, as $g(\mathbf{r}, p) \widetilde{\propto} \exp(-r/\xi(p))$. The correlation distance gives us a way to measure the characteristic radius of clusters. Notice that the definition refers only to sites in finite clusters, making $\xi(p)$ finite also for $p \geq p_c$.

Let us again look at the 1D example. Here, for a site displaced by $\mathbf{r}$ from a randomly chosen occupied site to be connected to it in 1D, all the sites in between need to be occupied, giving us

$$g(\mathbf{r}, p) = p^r = \exp\left(-\frac{r}{\xi(p)}\right),$$

where

$$\xi(p) = -\frac{1}{\ln(p)} \approx \frac{1}{p_c - p} \qquad \text{for } p \to p_c.$$

The result is again a power-law divergence near the percolation threshold, like in the previous cases. In general, we define the critical exponent ν via

$$\xi(p) \propto |p_c - p|^{-\nu} \qquad \text{for } p \to p_c, \tag{3}$$

where ν also only depends on the dimension of the lattice (see Table 2), with $\nu = 1$ in 1D.

Exponent	1D	2D	3D	4D	5D	6D
σ	1	36/91	0.45	0.48	0.49	1/2
β	/	5/36	0.41	0.64	0.84	1
γ	1	43/18	1.80	1.44	1.18	1
ν	1	4/3	0.88	0.68	0.57	1/2
d	1	91/48	2.53	3.06	3.54	4

Table 2. Critical exponents and the fractal dimension d of the infinite cluster for various lattice dimensions D in percolation theory. Values are taken from Ref. [5].

Such universal power-law divergences of observables at a critical value of a control parameter are analogous to second-order phase transitions in physics. However, for typical thermodynamic phase

transitions in physics, the control parameter is usually temperature (while the order parameter is, e.g., spontaneous magnetization), while in percolation the control parameter is instead the occupied site concentration p with the infinite cluster strength $P(p)$ serving as the order parameter. As we shall see, the analogy is not a coincidence, with renormalization theory providing a common explanation for the appearance of critical divergences near both thermodynamic and percolation transitions. Firstly, though, let us briefly look at the shape of clusters near a percolation transition.

2.3 Fractals

Interestingly, it turns out that large clusters exhibit *fractal geometry* near the percolation threshold. Specifically, the structure of the infinite cluster is self-similar close to the percolation threshold. Self-similarity is the defining property of fractals and means that if we take a smaller part of the cluster it will look, statistically speaking, similar to the whole cluster. Figure 2 shows a fractal infinite cluster.

Because of its fractal structure, the average number of occupied sites $N(L)$ in a side-length L subset of the infinite cluster scales differently with L than for non-fractal objects in a space with the same dimension $D \in \mathbb{N}$ as the lattice. Namely, while $N(L)$ still scales as a power law,

$$N(L) \propto L^d, \tag{4}$$

the exponent, which is called the *fractal dimension* $0 \leq d \leq D$, need not be an integer, while for a non-fractal object, it would simply be the integer dimensionality of that object.

We note that the scaling in Eq. (4) with the fractal dimension d is only exact at $p = p_c$, while for $p \approx p_c$ it only holds approximately on length scales below the correlation distance, $L < \xi(p)$, because for those the largest clusters still seem nearly infinite (and thus nearly fractal), while for $L > \xi(p)$ the number of sites $N(L)$ scales with the dimension D of the lattice.

It is also worth noting that the fractal dimension d is connected to the critical exponents for the characteristic cluster size s_ξ and correlation length ξ, σ and ν, respectively, via [5]

$$d = \frac{1}{\nu\sigma}.$$

We will show that the infinite cluster really has a fractal structure for $p = p_c$ using renormalization in the next section.

2.4 Renormalization

Renormalization is a collection of techniques used in quantum field theory, statistical field theory in thermodynamics, and in the theory of self-similar geometric structures. Here we will only show its application in the context of percolation theory as an approximation. The main idea of renormalization is to reduce the number of degrees of freedom of a system (e.g., in percolation, the number of lattice sites) by averaging out the behaviour on smaller scales in a way that maintains the large-scale behaviour of the system, at least to leading order. In percolation, this can be done in the following steps (illustrated in Figure 3 for a 2D lattice):

1. We divide the lattice into blocks of linear size b (in terms of the number of lattice sites).

2. We then look at each block; if most sites in the block are occupied then we mark the block as a whole as occupied, while if most sites are unoccupied we mark the whole block as unoccupied. Note that to apply this unambiguously, we need to have chosen an odd block size b in step 1, so that we always have a clear majority of occupied or unoccupied sites.

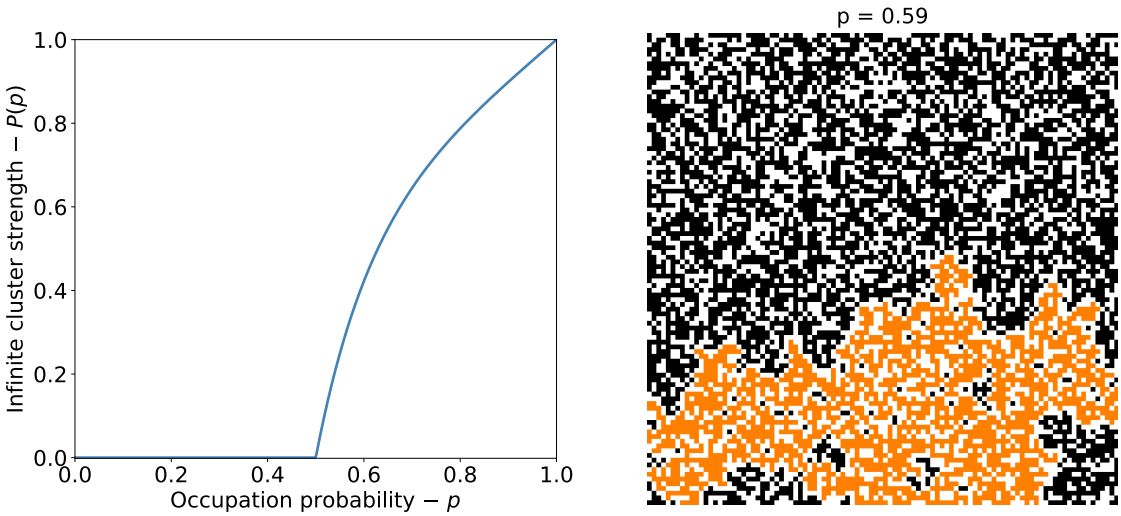


Figure 2. Left: Divergence of $P(p)$ for a 2D triangular lattice with $p_c = 0.5$, inspired by [5]. Right: Our numerical simulation for a 2D square lattice at $p = 0.59$, which is close to $p_c = 0.592746$ (Table 1). Here, occupied sites are depicted with black and unoccupied sites with white. The largest cluster is depicted with orange. We can observe the dense appearance of holes in the largest cluster decrease its fractal dimension to $d = 1.8875$, according to our numerical simulations, which is close to the theoretical value $d = 91/48 \approx 1.8958$ (see Table 2) [5], making it lower than the dimensionality $D = 2$ of the lattice itself.

Lastly, we restore the original lattice constant by dividing the new length with b. When we renormalized the lattice, the lattice parameters also need to be rescaled. What we obtain is again a percolation problem (on blocks), with a renormalized (block) concentration p' given by

$$p' = R_b(p) = \sum_{k=\lceil m/2 \rceil}^{m} \binom{m}{k} p^{m-k} (1-p)^k , \tag{5}$$

where $m = b^D$ is the number of all sites in a block and k counts the number of empty sites. Note that neglecting the site structure within a block in step 2 above makes this an approximation to the original problem, as even if two adjacent blocks are occupied, the site clusters within them might not actually link up on the site level. Nevertheless, the approximation should preserve large-scale structures, especially when clusters are large, $\xi \gg b$, i.e., close to the percolation threshold, $p \approx p_c$. Under renormalization, observable quantities also change (renormalize). For example, the correlation length ξ becomes

$$\xi' = \frac{\xi}{b}. \tag{6}$$

This process can be repeated as many times as we want.

With this in mind, we can first show that the infinite cluster has a fractal structure. We start by noting that after a sufficient (ideally, infinite) number of renormalization steps at $0 < p < 1$, we end up with a completely unoccupied lattice if in a single step $p' < p$ and with a completely occupied lattice if $p' > p$. However, for $p' = p$, which we call a fixed point p^* of the renormalization mapping Eq. (5) and is the renormalization approximation of the percolation threshold, $p_c \approx p^*$, percolation problem remains unchanged, meaning that its largest clusters are self-similar on all length scales, i.e., they are fractal. Furthermore, self-similarity also means that at the fixed point (i.e., at the percolation threshold) we have $\xi' = \xi$. Combined with Eq. (6), the only options for ξ are thus 0 and ∞, with the latter corresponding to the nontrivial fixed point p^* (i.e., ξ diverges at the percolation threshold, as assumed in Section 2.2).

The renormalization method can also give us approximate values for the critical exponents close to its fixed point. For example, the critical exponent ν can be determined by inserting the renormalization relation Eq. (6) into the ansatz Eq. (3) near the percolation threshold

$$\xi' = \frac{\xi}{b} \Longleftrightarrow |R_b(p) - p_\mathrm{c}|^{-\nu} = \frac{|p - p_\mathrm{c}|^{-\nu}}{b},$$

from which we obtain

$$\nu = \frac{\log(b)}{\ln\left(\frac{|R_b(p)-p_\mathrm{c}|}{|p-p_\mathrm{c}|}\right)} \approx \frac{\log(b)}{\ln\left(\frac{\mathrm{d}R_b(p_\mathrm{c})}{\mathrm{d}p}\right)}, \tag{7}$$

where in the last step we used the approximation

$$\frac{|R_b(p) - p_\mathrm{c}|}{|p - p_\mathrm{c}|} \approx \frac{|R_b(p) - R_b(p_\mathrm{c})|}{|p - p_\mathrm{c}|} \approx \frac{\mathrm{d}R_b(p_\mathrm{c})}{\mathrm{d}p} \qquad \text{for } p \to p_\mathrm{c}. \tag{8}$$

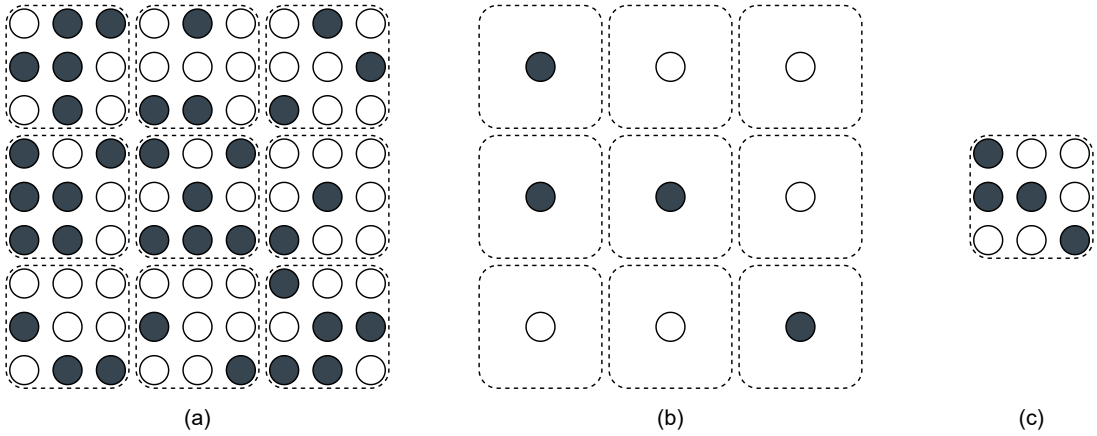


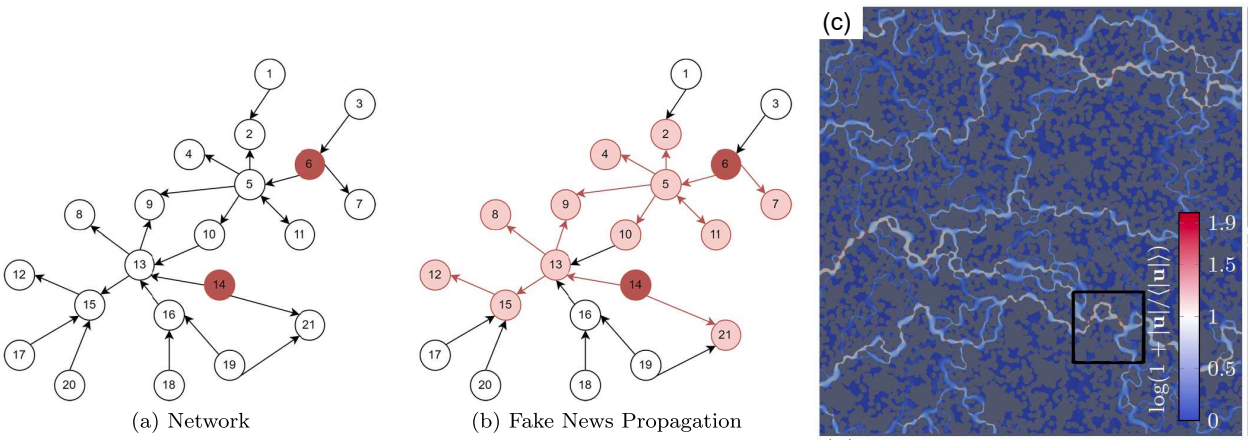
Figure 3. Example of renormalization for a 9×9 2D square lattice. (a) Division of the lattice into blocks with side length $b = 3$. (b) Blocks are assigned the correct (majority) state. (c) The original lattice constant is restored. Inspired by Ref. [5].

Finally, let us note that percolation theory can be generalized to cases other than just site percolation on periodic arrangements of sites. It can be used on non-periodic lattices or general graphs. Dense lattices can also mimic continuous media, as in example in Section 3.1 below. We can also introduce a correlation between the occupancies of different sites, where, for example, the likelihood of a site being occupied next to another occupied site is either higher or lower than if the other site was unoccupied. We can also study bond percolation, where sites are always present, but connections (bonds) between them are either occupied or not. This will be explained in more detail in Section 3.2, as it is useful for mapping percolation problems onto other thermodynamic (e.g., spin) systems. Such generalizations allow percolation theory to be used in a much larger set of systems, as we will see in the next section.

3. Some applications of percolation theory

3.1 Fluid flow in a 2D porous media at low Reynolds number

We can model a porous material by randomly placing blobs of a material, e.g., in the shapes of discs (in 2D) or spheres (in 3D) that we allow to overlap, surrounded by empty space. A 2D example of this is shown in Figure 4(c), where the material is shown in dark grey. The concentration parameter



(a) Network (b) Fake News Propagation

Figure 4. (a) A simple network with 20 nodes and connections between them. The arrows indicate the direction in which the news can spread. Red nodes are fake news spreaders. (b) The spreading of the fake news through the network, where pink nodes are the ones that received the fake news. Adapted from Ref. [7]. (c) A model of the porous media generated using overlapping discs. Colours represent the rescaled magnitude of the velocity $|u|/\langle|u|\rangle$ in a logarithmic scale, where $\langle|u|\rangle$ is the velocity magnitude averaged over the connected part of the sample. We can see small channels forming enabling flow through the medium. Adapted from Ref. [6].

$1-p$ is, in this case, the *packing fraction* ϕ, which gives the fraction of space covered by the material. A fluid is allowed to flow through the empty space around the blobs of material. At $\phi = \phi_c$ we have one spanning cluster in the porous medium. Below the threshold, there is a lot of empty space and there is a high flow rate. But when we approach ϕ_c the flow rate Q behaves like a power law and scales as an order parameter $Q \approx (\phi_c - \phi)^\alpha$ where $\alpha \approx 5/2$ [6]. At ϕ_c we have a transition from a connected (percolating) system to one dominated by bottlenecks and dead ends where there is very little flow through the medium. This is thus an example of correlated site percolation in a continuous (or at least very dense) medium.

3.2 Social media

We can also model the spread of fake news or rumours on social media using bond percolation on general graphs. We note that bond percolation has the same critical exponents as site percolation as it belongs to the same universality class as site percolation. Here, instead of having a periodic lattice with site occupation probability p and connections between nearest neighbours, we have a fixed number of always present nodes, while connections between arbitrary pairs of nodes have probability p to be present. These bonds represent friendships in the case of a social network. Connected random clusters are formed of people with connections present between them. Below the percolation threshold $p < p_c$ the clusters are finite, while for $p > p_c$ large, percolating clusters appear. We can then look at how misinformation spreads through such a social network. We randomly select some nodes to be the spreaders of misinformation, see Figure 4(a), that spread it to all nodes connected to it, see Figure 4(b). This goes on until all nodes in the same cluster have been reached. For $p < p_c$ the news thus only reaches small clusters of people while for $p > p_c$ the information spreads quickly through most of the social network. This model can be made more realistic by adding additional parameters or correlations, e.g., adding a probability that a node that receives misinformation will decide to pass it on.

3.3 2D bond percolation on a square lattice

Even though we usually do not get exact solutions for physical problems the power law divergences near p_c from Section 2. are universal [5]. Interestingly, cluster-size statistics for 2D bond percolation

on a square lattice with concentration p, can be mapped exactly onto the spin correlation functions of a ferromagnetic spin $\frac{1}{2}$ Ising model on a full square lattice at a finite temperature $T(p)$, with the percolation transition at p_c mapping exactly onto a thermodynamic phase transition (ferromagnetic ordering) at temperature T_c, as discovered by Kasteleyn and Fortuin [4].

4. Ising model

The Ising model is commonly used to model the appearance of spontaneous magnetization in magnets in the absence of an external field, as well as discrete systems more broadly. In magnetism, the magnetization in this model arises from exchange interactions between pairs of magnetic ions with the Hamiltonian

$$H = -J \sum_{\langle j,k \rangle} S_j^z S_k^z, \tag{9}$$

where J is the strength of exchange interactions and the sum is over pair of magnetic ions j and k with the z components S_j^z and S_k^z of spin, respectively. The exchange interactions originate from overlaps of the atomic orbitals, specifically from the Coulomb interaction and the Pauli exclusion principle. Typically, exchange interaction thus fall off rapidly with distance and thus we usually only need to consider the nearest neighbouring magnetic ions. J can have any sign, but we will focus on the case where $J > 0$, which leads to ferromagnetic ordering of spins at low temperatures. For $J < 0$, we can get antiferromagnetic ordering, while for $J = 0$ the spins do not interact and the system remains a paramagnet with no spontaneous magnetic ordering at any temperature. In what follows, we will take a look at the ferromagnetic Ising model on a randomly occupied 2D square lattice of magnetic ions with spin $\frac{1}{2}$.

4.1 Spin $\frac{1}{2}$ Ising model for random ferromagnets at low temperatures

The system we will simulate consists of a lattice of ferromagnetic ions with spin $\frac{1}{2}$ randomly mixed with non-magnetic impurities on a lattice. A random configuration of magnetic and non-magnetic ions can be represented as occupied and non-occupied sites in a percolation model, respectively, with the occupation probability p corresponding to the average concentration of magnetic ions. We will assume that the nearest neighbouring magnetic ions interact ferromagnetically via the Ising model, Eq. (9). Note that for spin $\frac{1}{2}$, there are only two possible spin projections, $+\frac{1}{2}$ (up) and $-\frac{1}{2}$ (down). At low temperatures, where thermal fluctuations are negligible (i.e., in a ground state of the model), nearest neighbouring magnetic ions will thus have the same orientation of magnetic moments. By extension, at low temperatures, each cluster in the percolation model will have a fixed value for the spin orientation. Since distinct clusters do not interact in the nearest-neighbouring Ising model, the spin orientations of separate clusters are thus chosen randomly and independently for each cluster.

On an infinite lattice, below the percolation threshold, $p < p_c$, we have infinitely many finite clusters and since the probability for either orientation of the magnetic moment is equally probable, the average magnetization is zero. At the percolation threshold, $p = p_c$, an infinite cluster forms at which point spontaneous magnetization becomes possible, at $p \geq p_c$ indicating a transition to a ferromagnetic ground state. When the infinite cluster first appears the total magnetization is due only to the infinite cluster since a finite proportion of magnetic ions belong to the infinite cluster, while the magnetization of magnetic ions in finite clusters (of which there are infinitely many) on average completely cancels out. Because of this, the spontaneous magnetization is proportional to the strength of the infinite cluster times the concentration of magnetic ions, i.e., to $pP(p)$, and onsets as an order-parameter power law at p_c with critical exponent β. On an infinite lattice at

$p = p_c$, the infinite cluster is a fractal that scales exactly with the fractal dimension d, while the correlation distance diverges, as a power law with critical exponent ν.

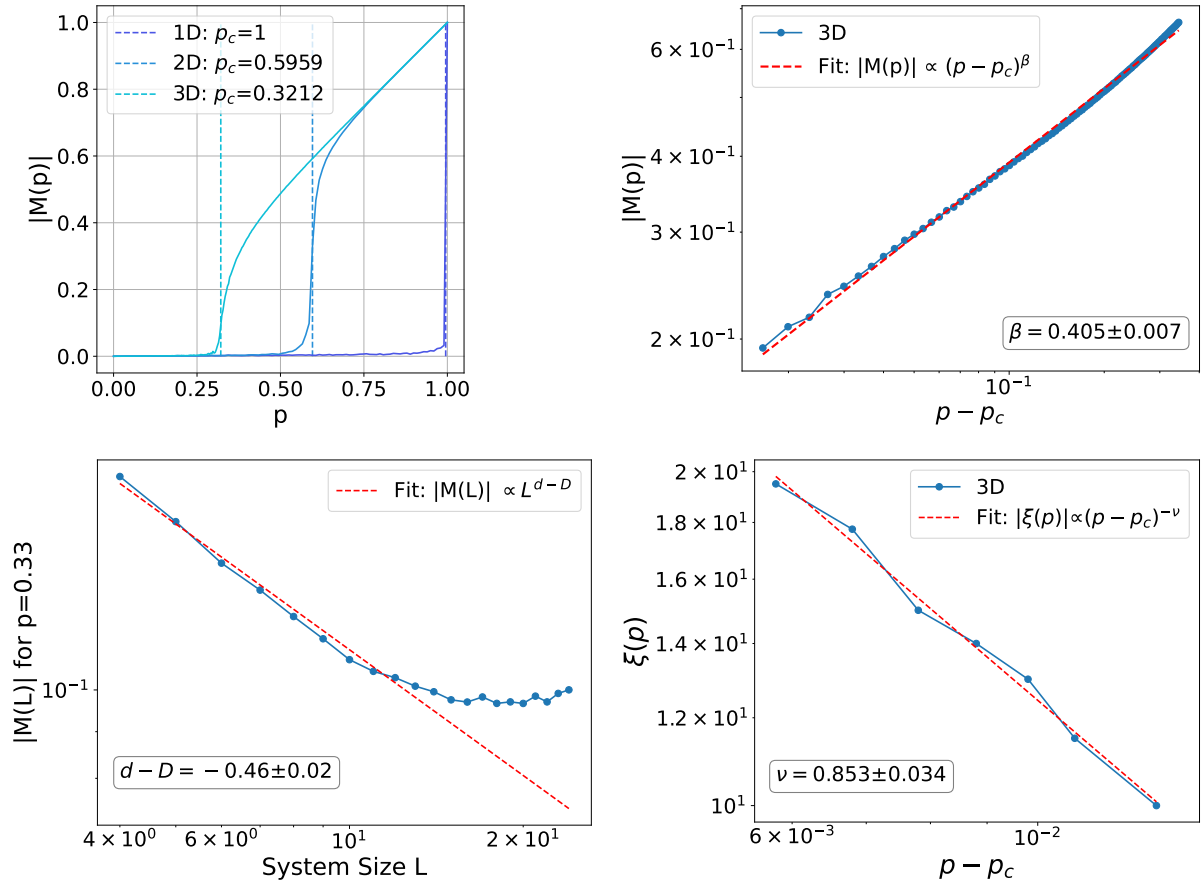


Figure 5. Upper left: Absolute spontaneous magnetization with respect to p. Upper right: Absolute spontaneous magnetization close to p_c with a fit from which we obtain β. Lower left: Absolute spontaneous magnetization with respect to system size L and a fit from which we obtain $A = d - D$. Lower right: Correlation distance close to p_c and a fit from which we obtain ν.

We have simulated a finite version of such a system (in Python) to test the values predicted by percolation theory. We were working with a 1D chain, a 2D square lattice, and a 3D simple cubic lattice. The algorithm was written so that it first filled a D-dimensional array of linear size L, representing the L^D lattice sites, with fixed random numbers between 0 and 1. Then, a site in the array is marked as occupied if the corresponding random number in the array is lower than p and unoccupied if it is larger. The algorithm then sorts all sites into clusters and randomly assigns a spin orientation to every cluster, where all the sites in one cluster have the same orientation. By increasing p from 0 to 1 in small increments (while keeping the same random numbers) and calculating the magnetization of the system for each value of p by summing over all the spins and dividing by the total number of occupied sites, we get a plot of spontaneous magnetization M (see the upper-left panel of Figure 5). From this plot, we can obtain the percolation threshold p_c. We did this by finding the inflection point of the magnetization curve as a function of p to have a robust criterion for p_c even on a finite lattice. We calculated the spontaneous magnetization for a 1D lattice with $L = 10^5$, a 2D lattice with $L = 500$, and $L = 50$ for a 3D lattice, while each value of magnetization was averaged over 100 independent simulations to reduce numerical noise. We then computed the critical exponents β [Eq. (2)] and ξ [Eq. (3)] and the fractal dimension d [Eq. (4)] for the 3D lattice. We obtained β by fitting $M(p) \propto (p - p_c)^\beta$ on the magnetization plot close to p_c (see

the upper-right panel of Figure 5). The fractal dimension d was found from a plot of magnetization at a constant p close to the percolation threshold but with a dependence on the lattice size L. In the lower-left panel of Figure 5, we can see a log–log plot where for small L the value steadily decreases and then stabilizes at larger L. The reason for the decrease is that $L < \xi(p)$ and the cluster has an approximately fractal structure until we reach some value L where this is not noticeable anymore. By fitting $|M(L)| \propto L^A$ we can obtain the difference $A = d - D$, since $M(L) = N(L)/(pL^D) \propto L^{d-D}$ by Eq. (4), and from that d. The value of L above which magnetization starts to saturate is the correlation distance $\xi(p)$. By finding $\xi(p)$ from plots of $M(L)$ for a few values of p close to the percolation threshold p_c, and plotting them versus $p - p_c$, we can extract the value of the critical exponent ν using Eq. (6) (see the lower-right panel of Figure 5).

We can now compare the values from the literature with the values from our simulations. For the percolation thresholds, We obtained $p_c^{1D} = 1 \pm 0.005$, $p_c^{2D} = 0.596 \pm 0.005$ and $p_c^{3D} = 0.321 \pm 0.005$, which coincides with the theoretical predictions of the percolation theory (see Table 2) for 1D and 2D, within error bars. The discrepancy in 3D, is likely due to the limitation of a finite lattice size. The critical exponent of the strength of the infinite cluster $\beta = 0.405 \pm 0.007$ from the simulations of the 3D lattice is also in good agreement with the one given by percolation theory (see Table 1). The fractal dimension of the infinite cluster that we obtain is $d = 2.54 \pm 0.02$ which is in good agreement with the theoretical value. Finally, the critical exponent for the correlation distance ξ near the percolation threshold we obtain is $\nu = 0.78 \pm 0.04$ which is relatively close to the one found in the literature (see Table 2). Overall, we find good agreement between our numerical simulations with theoretical expectations.

5. Conclusions

In this article, we described the basics of percolation theory, fractals, and renormalization. We looked at how percolation theory can be used to model fluid flow through a porous media, spreading of misinformation on social media and the thermodynamic phase transition of a ferromagnetic spin $\frac{1}{2}$. Finally, we compared the theoretical predictions for site percolation in 1, 2 and 3 dimensions with our numerical simulations and found that they are indeed in good agreement. Percolation theory can be used in many physical systems, while also being of high independent mathematical interest, and thus remains an active area of research in many different fields of science and mathematics.

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