QUANTUM CHAOS IN KICKED TOPS

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In classical systems, we often study two extreme regimes of motion: Regular integrable motion with many conserved quantities and irregular chaotic motion with exponential sensitivity to initial conditions. Characterizing chaos in quantum systems is more difficult. Studying quantum systems with a chaotic classical limit reveals the universal characteristics chaotic quantum systems usually have. It is conjectured that many of their statistical properties, such as the energy level spacing distribution, can be modeled with ensembles of random matrices, where the appropriate ensemble is determined only by the type of time-reversal symmetry of the system. This conjecture is numerically and experimentally well-tested. In this paper we demonstrate it on the example of quantum kicked tops. In contrast, the statistical properties of integrable quantum systems cannot be modelled by random matrices, making them special compared to the generic chaotic quantum systems.

KVANTNI KAOS V BRCANIH VRTAVKAH

Pri klasičnih sistemih tipično študiramo dva ekstremna režima gibanja: regularno integrabilno gibanje z veliko ohranjenimi količinami in neregularno kaotično gibanje z eksponentno občutljivostjo na začetne pogoje. Karakterizacija kaosa v kvantnih sistemih je težavnejša, lotimo se je s študiranjem kvantnih sistemov s kaotično klasično limito, kar nam razkrije univerzalne kvantne lastnosti, ki jih pričakujemo za kaotične sisteme. Znana je domneva, da veliko njihovih statističnih lastnosti, kot na primer porazdelitev razmikov med energijskimi nivoji, lahko modeliramo z ansambli naključnih matrik, kjer je pripadajoč ansambel določen le s tipom simetrije na obrat časa danega sistema. Opisana domneva je numerično in eksperimentalno dobro testirana, v tem članku jo demonstriramo na primeru kvantnih brcanih vrtavk. Statističnih lastnosti integrabilnih kvantnih sistemov ne moremo modelirati z naključnimi matrikami, zaradi česar jih štejemo kot posebne v primerjavi z generičnimi kaotičnimi kvantnimi sistemi.

1. Introduction

When studying nature from the point of view of classical physics, one comes across many systems displaying "regular" or "predictable" motion, the typical examples being a simple pendulum or the motion of a planet in a central potential. However, increasing the complexity of the system even slightly often leads to unpredictable or even seemingly random motion. One of the most well-known examples of this is the double pendulum, which takes different trajectories even if we try hard to replicate the same initial conditions. The situation can be mathematically described as a contrast between the regular motion of integrable systems and the irregular motion of chaotic systems.

Studying chaotic quantum systems has its roots in practical applications. In the 1950s, Wigner was the first to describe the properties of atomic nuclei with an ensemble of random matrices [1]. The system was too complicated to be solved exactly, which is why Wigner modelled it with essentially random matrices and produced accurate predictions about the highly excited nuclear energy spectra. Dyson expanded the idea by showing that Wigner's predictions are universal and depend only on the type of time-reversal symmetry of the given Hamiltonian [2]. After that, Wigner and Dyson's work was connected with classical chaos theory, with Bohigas, Giannoni and Schmit conjecturing that certain spectral characteristics of quantum systems with a chaotic classical limit can be described with ensembles of random matrices [3].

In this paper, we first state the main results of the classical study of chaos in Sec. 2. After that, periodically driven quantum systems are introduced in Sec. 3, an important example of which are quantum kicked tops, which have Hamiltonians dependent only on their angular momentum and will be used as examples in the following sections. In Sec. 4, the properties of integrable quantum systems are described. Furthermore, we introduce the time-reversal symmetry in Sec. 5, which is crucial for the understanding of the characteristics of quantum systems with a chaotic classical limit in Sec. 6.

2. Chaos in classical systems

Let us consider a classical Hamiltonian system with f degrees of freedom

$$H(q_1, q_2, \ldots, q_f, p_1, p_2, \ldots, p_f) = H(\mathbf{x}),$$

where q_i are generalized coordinates and p_i are generalized momenta. Its motion can be fully described in $2f$-dimensional phase space. The theory of dynamical systems strives to mathematically describe and classify dynamical systems based on the “regularity” of their motion. We often describe the motion of Hamiltonian systems as belonging into one of the two extreme regimes with radically different behavior: Chaotic and integrable motion.

Chaotic motion can be characterized by its exponential sensitivity to initial conditions. More specifically, given two distinct initial conditions $\mathbf{x}_1(t=0)$ and $\mathbf{x}_2(t=0)$ and evolving them with Hamilton's equations to $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$, we expect

$$|\mathbf{x}_2(t) - \mathbf{x}_1(t)| \approx |\mathbf{x}_2(0) - \mathbf{x}_1(0)| e^{\lambda t}, \tag{1}$$

for long times t and small $|\mathbf{x}_2(0) - \mathbf{x}_1(0)|$. In Eq. (1), $\lambda > 0$ is the Lyapunov exponent and is different for every system [4]. This means that even a little perturbation of the initial conditions leads to radically different trajectories, which one might intuitively describe as irregular or even random motion (even though the system is completely deterministic).

Non-chaotic systems do not display exponential sensitivity to initial conditions and therefore do not have a positive Lyapunov exponent (for conservative systems in this case $\lambda = 0$). In addition to that, integrable Hamiltonian systems have f independent conserved quantities with vanishing Poisson brackets. Furthermore, it can be shown that (under mild assumptions) the motion of an integrable system always lies on a surface topologically equivalent to a torus [4]. Because of that, we usually intuitively describe integrable motion as regular motion.

Most sufficiently complex dynamical systems, in nature and theoretically, are chaotic [4], which is why we say that chaotic motion is generic and integrable motion is special.

3. Periodically driven quantum systems

We would like to study the properties of quantum systems with integrable and chaotic classical limits, so we need sufficiently complex quantum systems. One of the simplest classes of systems in which it is easy to generate chaos are periodically driven systems with time-dependent Hamiltonians of the form

$$H(t) = H_0 + V \sum_{n=-\infty}^{\infty} \delta(t - n\tau), \tag{2}$$

where δ is the Dirac delta function, H_0 and V are some operators and τ is the period. These systems thus evolve according to H_0 and are kicked in an infinitesimally short time every period τ.

When studying such systems, it is natural to focus on the so-called Floquet operator F, which propagates the system by one period τ

$$\psi(t + \tau) = F\psi(t).$$

Given a Hamiltonian of the form (2), it can be shown that its Floquet operator is [5]

$$F = \exp(-iV/\hbar) \exp(-iH_0\tau/\hbar). \tag{3}$$

This can be understood intuitively: Every period starts with evolution according to H_0 ($\exp(-iH_0\tau/\hbar)$) and is followed by a kick ($\exp(-i\lambda V/\hbar)$). The kick happens in an infinitesimally short time, which means that it is independent of the H_0 contribution and can be factored in its own exponent.

Since Floquet operators are propagators for a fixed time, they are unitary. They can therefore be diagonalized and their eigenvalues take the form of

$$F\Phi_j = \exp(-i\phi_j)\Phi_j,$$

where $\phi_j \in \mathbb{R}$ are referred to as the quasi-energies of the system.

3.1 Kicked quantum tops

Quantum tops are quantum systems with Hamiltonians dependent only on their angular momentum $\mathbf{J}$. Their Hilbert space is $2(j+1)$ dimensional, where $\mathbf{J}^2 |jm\rangle = j(j+1) |jm\rangle$. The simplest example of a quantum top is

$$H = \hbar\tilde{\alpha}_3 J_z,$$

where $\hbar$ is the Planck constant and $\tilde{\alpha}_3$ is some constant. It is well-known that the expectation value of angular momentum for such Hamiltonians precesses about the z-axis, analogous to the movement of a classical spinning top. Usually, this is studied for magnetic dipoles in a magnetic field and is known as the Larmor precession [6].

Kicked quantum tops are quantum tops with an additional kicking term similar to Eq. (2). A simple example is

$$H(t) = \frac{\hbar\alpha_3}{\tau} J_z + \frac{\hbar\beta_3}{2j+1} J_z^2 \sum_{n=-\infty}^{\infty} \delta(t - n\tau),$$

where α_3 and β_3 are some constants. According to Eq. (3), its Floquet operator is

$$F = \exp\left(-i\frac{\beta_3}{2j+1} J_z^2\right) \exp(-i\alpha_3 J_z).$$

It can be shown that the classical limit of such tops is integrable [7]. To study systems with chaotic classical limits, we will consider more general tops with Floquet operators of the form

$$\begin{aligned} F = &\exp\left(-i\frac{\beta_3}{2j+1} J_z^2\right) \exp(-i\alpha_3 J_z)\cdot \\ &\exp\left(-i\frac{\beta_2}{2j+1} J_y^2\right) \exp(-i\alpha_2 J_y)\cdot \\ &\exp\left(-i\frac{\beta_1}{2j+1} J_x^2\right) \exp(-i\alpha_1 J_x), \end{aligned} \tag{4}$$

which are characterized by constants $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$. Physically, we can imagine that Floquet operators of the form (4) describe systems with a 6-step period schematically shown in Fig. 1.

4. Integrable quantum systems

Formulating a good definition of integrability in a quantum sense is harder than its classical counterpart from Sec. 2. An analogous definition that requires integrable quantum systems to have f mutually commuting conserved quantities, where f is the number of degrees of freedom, turns out to be too naive since projectors to eigenspaces can always fulfill this role [8]. In many-body systems, this can typically be circumvented by requiring that the conserved quantities are local [9]. Generally, however, different definitions are considered in different cases, one of the simplest (but

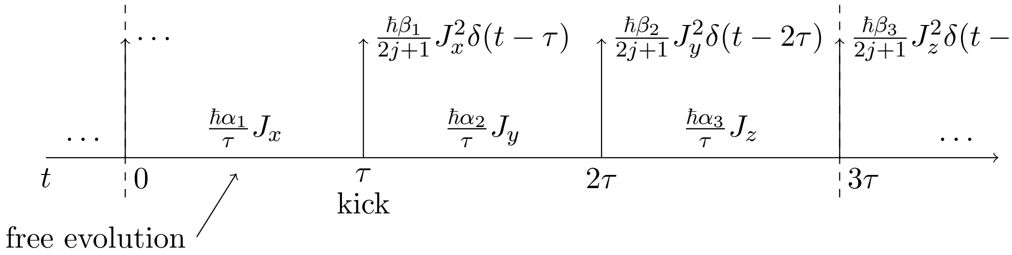

Figure 1. Diagram of the general quantum kicked top with the Floquet operator from Eq. (4). The first period is $t \in (0, 3\tau]$ and consist of 3 different free evolution operators, each followed by its own kick.

again often inadequate) being that integrable systems are systems that are exactly solvable [8]. An important feature of integrable quantum systems that every definition should imply is a large number of conserved quantities. Therefore, their Hamiltonians or Floquet operators reduce to a block diagonal form with a large number of irreducible blocks indexed by the conserved quantities.

The kicked top [Eq. (4)] with $\boldsymbol{\alpha} = (0, 0, 1)$ and $\boldsymbol{\beta} = (0, 0, 1)$ is exactly solvable (its eigenbasis is equal to the eigenbasis of J_z) and can thus be said to be integrable[1]. When studying systems with a chaotic classical limit later, we will be interested in how the (quasi) energy levels change when varying one of their parameters. If we vary a parameter of an integrable system, for example $\boldsymbol{\alpha} = (0, 0, \xi)$, the quasi-energy levels change independently in each irreducible block of the Floquet operator and readily cross each other. This seemingly messy behavior is shown in Fig. 2.

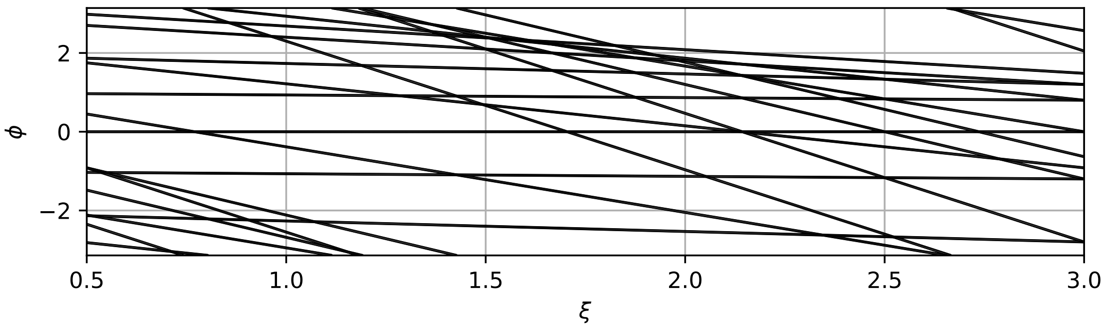

Figure 2. Quasi-energy levels for the quantum top [Eq. (4)] with $\boldsymbol{\alpha} = (0, 0, \xi)$, $\boldsymbol{\beta} = (0, 0, 1)$. This top is integrable [5, 7]. $j = 7$.

4.1 Energy level clustering

Because (quasi) energy levels of integrable quantum systems seem to readily cross, we are compelled to study the (quasi) energy spacing distribution, which can be defined as

$$P(S) = \langle \delta(S - \Delta E) \rangle \,, \tag{5}$$

where δ is the Dirac delta function, ΔE is the spacing between neighboring (quasi) energy levels and $\langle \bullet \rangle$ denotes an average over all ΔE.

Using the semi-classical Einstein-Brillouin-Keller approximation [10], it can be shown that integrable quantum systems usually exhibit a Poissonian level spacing distribution [5]

$$P(S) = e^{-S}.$$

[1]It also exhibits other features that are typical for integrable systems [5, 7], some of which we will see in this article.

The Poisson statistic has a peak at $S = 0$, which means that the energy levels in an integrable quantum system tend to cluster. This is demonstrated on the example of the integrable top from Sec. 4. in Fig. 3, where we see good agreement with the theory.

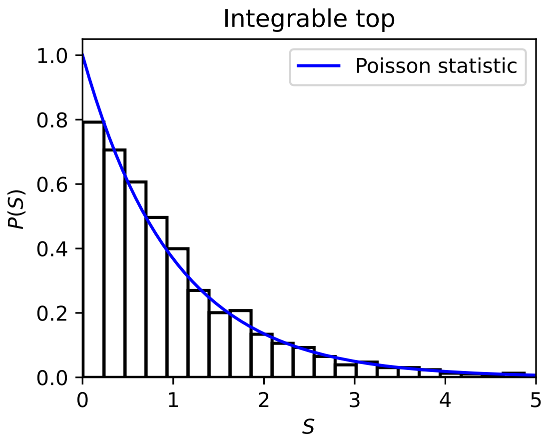

Figure 3. Numerical calculation of the quasi-energy spacing distribution for the quantum top [Eq. (4)] with $\boldsymbol{\alpha} = (0, 0, 1)$, $\boldsymbol{\beta} = (0, 0, 1)$ compared with the Poisson distribution. This top is integrable [7]. $j = 10^3$.

There are exceptions to this rule however, one of the simplest being the quantum harmonic oscillator, which has constant energy spacing. The exact assumptions needed for a Poissonian level spacing distribution are known [5], but are too technical for this paper.

5. Time-reversal symmetry

Before turning to quantum systems with chaotic classical limits, we take a brief detour to define time-reversal symmetry, which will be crucial for their classification.

5.1 Time-reversal operator

The Schrödinger equation for a spinless particle with mass m and momentum operator $\mathbf{p}$ in a real potential V

$$i\hbar \frac{\partial \psi(\mathbf{x}, t)}{\partial t} = H\psi(\mathbf{x}, t) = \left[\frac{\mathbf{p}^2}{2m} + V(\mathbf{x})\right]\psi(\mathbf{x}, t), \qquad V(\mathbf{x}) = V^*(\mathbf{x}), \tag{6}$$

where ψ is the wave function, is invariant to time reversal. More specifically, if $\psi(\mathbf{x}, t)$ is its solution, then so is $\psi^*(\mathbf{x}, -t)$, where * denotes complex conjugation [6]. In this case, we can thus define the time-reversal operator T as

$$T\psi(\mathbf{x}) = K\psi(\mathbf{x}) = \psi^*(\mathbf{x}), \tag{7}$$

where K denotes the complex conjugation in the $\mathbf{x}$ representation.

Time-reversal operators can be generalized. For T to be a time-reversal operator, we demand:

$$\langle T\psi | T\phi \rangle = \langle \phi | \psi \rangle \qquad \text{(antiunitarity)}, \tag{8}$$

$$T^2 = \pm 1. \tag{9}$$

Antiunitarity [Eq. (8)] is needed because of the i factor on the left-hand side of the Schrödinger equation [Eq. (6)]. The second requirement [Eq. (9)] is physically reasonable because we want that double time reversal reproduces the same wave function, up to a phase factor.[2] It can be easily checked that T from Eq. (7) meets both of these requirements [5].

[2] In fact, Eq. (9) can be replaced by $T^2 = \alpha$, $|\alpha| = 1$ and $T^2 = \pm 1$ then follows from Eq. (8).

Another well-known example of time-reversal is time-reversal for spin 1/2 particles, which is equal to

$$T = i\sigma_y K, \tag{10}$$

where σ_y is the y Pauli matrix [6]. In this case $T^2 = -1$.

5.2 Canonical transformations

Hamiltonians are Hermitian operators and are thus represented by Hermitian matrices in any orthonormal basis. We define (quantum) canonical transformations as basis transformations, which preserve the Hamiltonian's hermiticity and its eigenvalues. Symmetries play a central role in determining all the possible canonical transformations. In this paper, we focus on time-reversal symmetries and assume that our Hamiltonians do not have additional geometric (unitary) symmetries. If they do, they must be studied separately in each irreducible block induced by geometric symmetries [5].

It is well-known that unitary transformations conserve eigenvalues. They also conserve hermiticity, since

$$\left(UHU^\dagger\right)^\dagger = \left(U^\dagger\right)^\dagger H^\dagger U^\dagger = UHU^\dagger.$$

In fact, it can be shown that only unitary transformations fulfill the requirements of canonical transformations [5]. We, therefore, say that the class of canonical transformations for a general Hamiltonian in N-dimensional Hilbert space is the group of unitary matrices $U(N)$.

Time-reversal symmetries can restrict the class of canonical transformations. Let us have a time-reversal operator T with $T^2 = 1$. We say that the Hamiltonian has a time-reversal symmetry if

$$[H, T] = 0 \qquad \iff \qquad THT = H. \tag{11}$$

In this case, we can construct a time-reversal invariant basis $\{\psi_i\}$. For any vector ϕ_1, we define

$$\psi_1 = a_1\phi_1 + Ta_1\phi_1,$$

for any $a_1 \in \mathbb{C}$. It is clear that $T\psi_1 = \psi_1$. Furthermore, for any ϕ_2 orthogonal to ψ_1, we define

$$\psi_2 = a_2\phi_2 + Ta_2\phi_2,$$

for any $a_2 \in \mathbb{C}$. Here again $T\psi_2 = \psi_2$. But also

$$\langle\psi_2|\psi_1\rangle = a_2^* \langle\phi_2|\psi_1\rangle + a_2 \langle T\phi_2|\psi_1\rangle = a_2 \left\langle T\psi_1\middle|T^2\phi_2\right\rangle = a_2 \langle\psi_1|\phi_2\rangle = 0,$$

where we have used antiunitarity [Eq. (8)] and $\langle\psi_1|\phi_2\rangle = 0$.

By iterating the above procedure, we construct the basis $\{\psi_i\}$, which is thus orthogonal and a_i can be chosen in such a way that it is orthonormal. Moreover, the Hamiltonian in this basis is real

$$H_{ij} = \langle\psi_i|H|\psi_j\rangle = \langle TH\psi_j|T\psi_i\rangle = \langle\psi_i|THT^2|\psi_j\rangle^* = \langle\psi_i|THT|\psi_j\rangle^* = H_{ij}^*$$

Here we have again used the antiunitary [Eq. (8)], Eq. (11), $T^2 = 1$ and $T\psi_i = \psi_i$.

We have shown that given a Hamiltonian with time-reversal symmetry with $T^2 = 1$, we can always find a basis in which H_{ij} are real without diagonalization. It is therefore fair to say that such Hamiltonians are generically real. Because of this, their class of canonical transformations can be reduced to real unitary operators; in the case of an N-dimensional Hilbert space these are precisely the orthogonal matrices from the group $O(N)$.

Similarly, it can be shown that canonical transformations for Hamiltonians with time-reversal symmetry with $T^2 = -1$, reduce to the group of symplectic matrices [5]

$$Sp(2N) = \left\{S \in \mathbb{R}^{2N\times 2N} | SZS^T = Z\right\},$$

where Z is a block-diagonal matrix with diagonal 2×2 blocks equal to $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$.

We have now completely classified the canonical transformations of Hamiltonians based on their time-reversal symmetry. This is called the Wigner-Dyson classification and is summarized in Table 1. The simplest examples are the spinless time reversal from Eq. (7) and spin 1/2 time reversal from Eq. (10).

Time-reversal symmetry	Canonical transformations	Random matrix ensemble
No	$U(N)$	Gaussian unitary ensemble (GUE)
Yes, $T^2 = 1$	$O(N)$	Gaussian orthogonal ensemble (GOE)
Yes, $T^2 = -1$	$Sp(2N)$	Gaussian symplectic ensemble (GSE)

Table 1. Wigner-Dyson classification of Hamiltonians without geometric symmetries based on their time-reversal symmetry and the corresponding random matrix ensemble (Sec. 6.3).

5.3 Time-reversal for Floquet operators

It can be shown that the Floquet operators corresponding to time-reversal invariant Hamiltonians [Eq. (11)] behave under time-reversal as

$$TFT^{-1} = F^{\dagger}, \tag{12}$$

which can be intuitively understood [5]. Since F is a unitary operator $F^{\dagger} = F^{-1}$, Eq. (12) implies that the time-reversed Floquet operator is equal to its inverse. In other words, time-reversal of an operator that evolves the system by one period forward is an operator that evolves the system by one period backward.

Analogous to Sec. 5.2, we can now define the canonical transformations of Floquet operators as transformations, which preserve the unitarity and the eigenvalues of a Floquet operator. Moreover, by employing similar arguments, it can be shown that classes of canonical transformations for Floquet operators are the same as for Hamiltonians [5]. This means that Floquet operators without time-reversal symmetry have a canonical transformation class $U(N)$, those with $T^2 = 1$ time-reversal symmetry have $O(N)$ and those with $T^2 = -1$ have $Sp(2N)$.

6. Quantum systems with chaotic classical limits

In contrast to integrable quantum systems, it is not obvious how to exactly define a chaotic quantum system [5]. The first idea might be to define a chaotic quantum system as a system, which is chaotic in its classical limit. However, this is not adequate, since we want to have a purely quantum definition, which could be applied to quantum systems without a classical limit, such as spin chains. In this section, we examine some properties of quantum systems with chaotic classical limits, which gives us an idea of what a purely quantum definition of chaos could look like.

To use as examples, we pick 2 tops

- the unitary chaotic top: $\boldsymbol{\alpha} = (1, 2, 3), \boldsymbol{\beta} = (1.2, 1.5, 1)$,
- the orthogonal chaotic top: $\boldsymbol{\alpha} = (0, 2, 3), \boldsymbol{\beta} = (0, 1.5, 1)$.

It can be shown that both of them have a chaotic classical limit. Furthermore, the unitary top has the class of canonical transformations $U(N)$ and the orthogonal top has the class of canonical transformations $O(N)$ [7, 11]. Simpler tops with desired properties do exist, but they possess additional geometrical symmetries, which are out of the scope of this paper. Tops with $Sp(2N)$ canonical transformations also exist [5].

6.1 Energy level repulsion

In contrast to the behavior of integrable systems in Sec. 4, the (quasi) energy levels of one parameter Hamiltonians $H(\xi)$ or Floquet operators $F(\xi)$ without conserved quantities (nonintegrable) typically do not cross with varying ξ [3, 12]. Figure 4 illustrates the described phenomena on the quasi-energy spectrum of a generalized one-parameter version of the unitary chaotic top from Sec. 6. Notice that there is no crossing of energy levels (if the picture is zoomed in enough) and upon close encounter, the energy level curves even seem to repel each other.

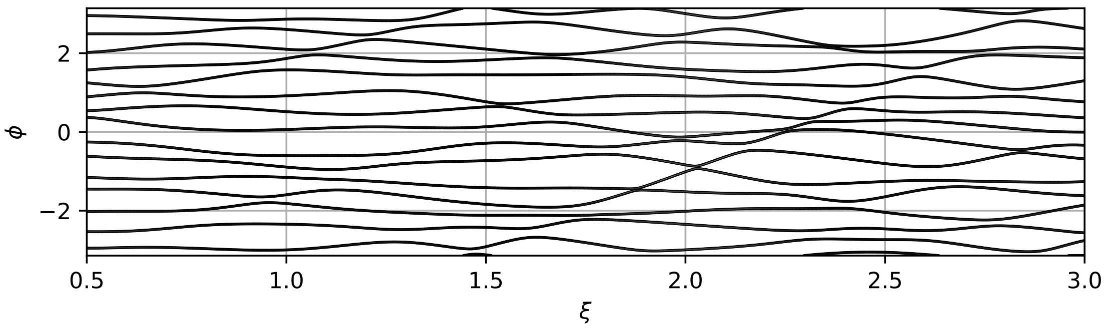

Figure 4. Quasi-energy levels for a quantum top [Eq. (4)] with $\boldsymbol{\alpha} = (1, 2, 3)$, $\boldsymbol{\beta} = (1.2, 1.5, \xi)$ and $j = 7$.

Let us study level repulsion in more detail. Upon close encounter of two energy levels, they may be treated with a nearly degenerate perturbation theory, leading to an effective 2×2 Hamiltonian

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix},$$

where $H_{11}, H_{22} \in \mathbb{R}$ and $H_{12} \in \mathbb{C}$ to ensure hermiticity. The eigenvalues of this matrix are

$$E_\pm = \frac{1}{2}(H_{11} + H_{22}) \pm \sqrt{\frac{1}{4}(H_{11} - H_{22})^2 + \mathrm{Re}(H_{12})^2 + \mathrm{Im}(H_{12})^2}. \tag{13}$$

To get a level crossing, we require $E_+ = E_-$, which is only possible when the discriminant vanishes. The discriminant is the sum of 3 non-negative terms, which thus all must vanish. Therefore, for systems with generic enough Hamiltonians, we expect that a crossing happens only exceptionally. As discussed in Sec. 2, complex enough generic classical systems are chaotic, which is why we can expect level repulsion in their quantum version [3, 12].

The number of parameters that must be controlled to get a level crossing [i.e. for the discriminant in Eq. (13) to vanish] is called the codimension n of the crossing. As discussed in the previous paragraph, this would be $n = 3$ for general Hamiltonians, which are in the unitary Wigner-Dyson class. Hamiltonians in the orthogonal class are generically real, which means that the imaginary term in Eq. (13) always vanishes, leading to $n = 2$. In general, it can be shown [5]

$$n = \begin{cases} 2 & \text{orthogonal class} \\ 3 & \text{unitary class} \\ 5 & \text{symplectic class} \end{cases}.$$

By treating Floquet operators in a similar way, the same values of n are acquired for quasi-energy level crossings [5].

6.2 Energy level spacing distribution

The codimension of a given level crossing plays a crucial role in the distribution of (quasi) energy level spacings defined in Eq. (5). According to Eq. (13), the energy spacing (the discriminant) can be written as

$$\Delta E = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\mathbf{x}^2},$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ are some parameters and n is the codimension of the crossing. By writing out the average in Eq. (5) as an integral with a suitable but unknown weight $W(\mathbf{x})$

$$P(S) = \int \mathrm{d}^n\mathbf{x}\, W(\mathbf{x})\delta(S - \sqrt{\mathbf{x}^2}) = S^{n-1} \int \mathrm{d}^n\mathbf{x}\, W(S\mathbf{x})\delta(1 - \sqrt{\mathbf{x}^2}),$$

where we have preformed $\mathbf{x} \to S\mathbf{x}$ and used the well-known property $\delta(\alpha x) = \delta(x)/|\alpha|$.

Provided that $W(\mathbf{x})$ is finite and non-zero at $\mathbf{x} = 0$, we thus showed that

$$P(S) \propto S^{n-1} \qquad \text{when} \qquad S \to 0 \tag{14}$$

with the general form of $P(S)$ unknown.

6.3 Random matrix theory

To better understand the energy spacing distribution $P(S)$, we need a more general model of a quantum system with a chaotic classical limit. As discussed in Sec. 2, chaos in sufficiently complex classical systems is generic, which leads us to an idea to define an ensemble of random matrices and compare their behavior to the behavior of the studied system. This is the idea of random matrix theory (RMT).

The mathematics of random matrix theory is too technical for this paper, which is why we only state the general ideas and the main results. We can define 6 RMT ensembles, 3 Gaussian ensembles and 3 circular ensembles. The Gaussian ensembles contain Hermitian matrices, so they apply to Hamiltonians, and circular ensembles contain unitary matrices, thus being relevant for Floquet operators. Both kinds then come in 3 forms, corresponding to their canonical transformations; unitary, orthogonal or symplectic. Table 1 summarizes the Gaussian RMT ensembles and the corresponding Hamiltonian Wigner-Dyson classes.

To define the ensembles, we need their probability density $P(M)$ for a matrix M. The probability densities are exactly determined by 3 requirements:

- Matrices M are Hermitian (Gaussian ensembles) or unitary (circular ensembles),
- $P(M)$ is invariant under the chosen canonical transformation (unitary, orthogonal or symplectic),
- The probability densities for matrix elements $P(M_{ij})$ are uncorrelated.

For Gaussian ensembles, this results in $P(H) = C \exp\bigl(-A \operatorname{tr} H^2\bigr)$, but with different constants A, C, hence their name [5].

Obtaining the probability densities, one can calculate many quantities by averaging over the whole ensemble. For the Gaussian ensembles of 2×2 matrices, the energy (eigenvalue) spacing distributions can be calculated [5]

$$P_{\mathrm{Wigner}}(S) = \begin{cases} \frac{\pi}{2} S \exp\left(-S^2\pi/4\right) & \text{ortgohonal ensemble} \\ \frac{32}{\pi^2} S^2 \exp\left(-S^2 4/\pi\right) & \text{unitary ensemble} \\ \frac{2^{18}}{3^6\pi^3} S^4 \exp\left(-S^2 64/9\pi\right) & \text{symplectic ensemble} \end{cases} \tag{15}$$

and are commonly referred to as Wigner surmises. Surprisingly, the energy spacing distributions for arbitrary dimensional Gaussian and circular ensembles differ only slightly from the Wigner surmises [5], which is why most often in practice, the Wigner surmises are used regardless of the dimensionality of the ensemble. The Wigner surmises, along with the Poisson statistic $P(S) = e^{-S}$, are shown in Fig. 5. Notice that their behavior at $S \to 0$ corresponds to Eq. (14).

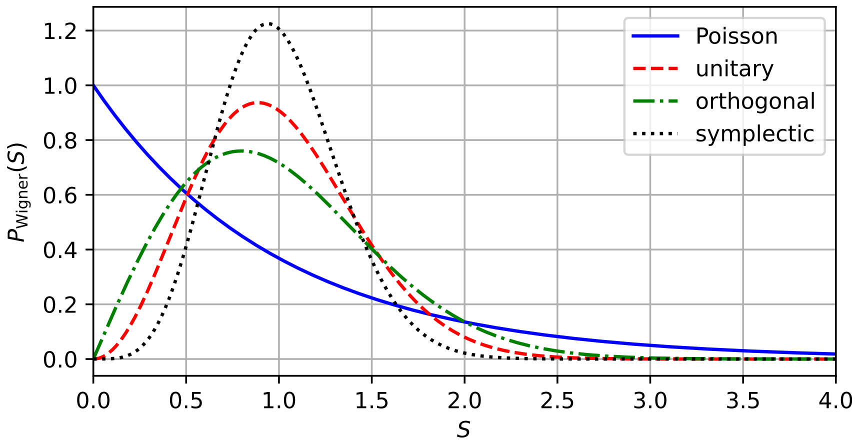

Figure 5. The Wigner surmises defined in Eq. (15) and the Poisson statistic.

Finally, we can state the quantum chaos conjecture: Quantum systems with a chaotic classical limit display (quasi) energy level spacing distributions predicted by the corresponding RMT ensemble. This was first stated in a slightly weaker form by Bohigas, Giannoni and Schmit [3]. Although there is currently no rigorous proof of the conjecture, there exists vast numerical and experimental evidence [5, 7, 13]. Some of the quantum systems most faithful to the conjecture are the already mentioned kicked tops. Their energy spacing distributions are shown and compared with Wigner surmises in Fig. 6, where we see good agreement. Physically, this means that quantum systems with a chaotic classical limit behave essentially as random matrices, with their statistical properties depending only on their Wigner-Dyson class and thus on their type (or lack of) time-reversal symmetry. This is why we often refer to Wigner-Dyson classes as universality classes of quantum systems.

The reverse of the quantum chaos conjecture was shown not be true: There exist integrable quantum systems exhibiting energy level spacing distributions predicted by RMT, for example specially constructed many-body bosonic systems [14]. Because of that, an RMT predicted level spacing distribution cannot be a defining feature of chaotic quantum systems, but it can be a good indicator that we are dealing with a system with a chaotic classical limit. In systems without a classical limit, we usually test more indicators that were not mentioned before claiming that a system is chaotic [5].

7. Conclusions

In conclusion, we explained the well-tested conjecture that quantum systems with a chaotic classical limit can be described by ensembles of random matrices. One of the quantities well predicted by

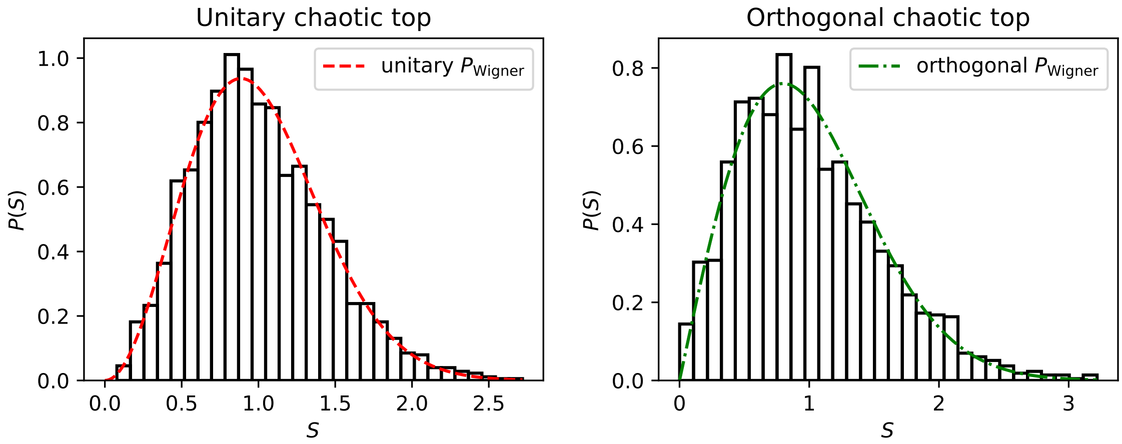

Figure 6. Numerical calculation of quasi-energy spacing distributions for chaotic kicked tops defined in Sec. 6. compared with Wigner surmises from Eq. (15). $j = 10^3$.

the random matrix theory is the energy level spacing distribution. The statistical level spacing behavior is thus universal and the universality class of a system is determined only by its time-reversal symmetry. Integrable quantum systems cannot be modelled with random matrices and their energy level spacing distributions usually follow a Poisson statistic.

We have numerically shown that kicked tops are faithful to the conjecture. Recently, this was also tested experimentally, with kicked tops realized as cesium atoms in a magnetic field [13]. This adds to other already existing experimental evidence, mostly in the field of nuclear [15] and atomic physics [16]. Quantum chaos is still interesting theoretically, with current research focusing on many-body systems [17], spin chains [18] and the relationship with transport phenomena [19], to name just a few.

This paper has focused on energy level spacing distributions, which are one of the most often studied characteristics of chaotic quantum systems. We left out a few technical details unimportant for kicked tops, mainly the treatment of additional geometric symmetries and inhomogeneous spectra with spectral unfolding. Other important quantities to study in quantum chaos include correlation functions and the related notion of the spectral form factor. In special cases, for example, when treating superconducting or relativistic fermions, the Wigner-Dyson classification is incomplete and 7 additional universality classes are needed [5].

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